MATHEMATICS TEACHING toolkit

ISSUES IN THE TEACHING OF MATHEMATICS

Teaching with the Big Ideas in Mathematics

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THE BIG PICTURE

BIG IDEAS IN MATHEMATICS

The notion of *Big Ideas* in mathematics education is not new. Many years ago, Bruner (1960) noted that knowledge "acquired without sufficient structure is knowledge that is likely to be forgotten" (p. 31). Charles (2005) defined a 'Big Idea' in mathematics as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p. 10). For example, while most agree that *Place Value* is a Big Idea for this purpose (e.g., Askew, 2013; Charles, 2005; Ma, 1999; Siemon, 2006; Van de Walle et al., 2010), beyond this there is little agreement about what these Big Ideas are or how they are best represented to support the teaching and learning of school mathematics. Big Ideas need to be both mathematically important and pedagogically appropriate (Askew, 2013; Siemon et al., 2012).

Big Ideas in mathematics provide an organising framework for teachers to think about their task as teachers of mathematics. When teachers are aware of these ideas and their role in the 'mathematical landscape', they are able to 'look backwards' and plan their teaching based on an understanding of where learners are in the landscape. They are also able to anticipate students' mathematical futures, to 'see', 'hear', and 'act' on the possibilities afforded by the insights offered by students as they grapple with the mathematics of the present.

This monograph considers the issue of Big Ideas in school mathematics in terms of the following two questions:

- **What are Big Ideas in school mathematics and why are they important?**
- **What implications do Big Ideas have for the teaching and learning of mathematics?**

KEY TERMS AND DEFINITIONS

Algorithm

According to the Oxford Dictionary, an algorithm is a process or set of rules to be followed in calculations or other problem-solving operations, especially by a computer.

Big Ideas

Views about what constitute a Big Idea vary considerably (See Appendix 1) and a single definition will not be provided here beyond Askew's (2015) observation that they are "mathematically big, conceptually big, and pedagogically big" (p. 13). That is, they have a basis in mathematics but also in how mathematics is taught and learnt (Ball & Bass, 2009). Other terms that are frequently used in relation to Big Ideas are briefly described below.

Concept

Usually refers to the idea of a class of objects or attributes (e.g., chair, yellow). This term used in association with Big Ideas suggests a cluster of related ideas and processes. Clarke's (1997) "working definition of a 'concept' is a big idea that helps us makes sense of, or connect, lots of little ideas. Concepts are like cognitive file folders. They provide us with a framework or structure within which we can file an almost limitless amount of information. One of the unique features of these conceptual files is their capacity for cross-referencing" (p. 94).

Connections

Used in most discussions of Big Ideas and what itmeans to understand mathematics (e.g., Charles, 2005; Hiebert & Carpenter, 1992; Vale et al, 2011). For instance, Van de Walle et al. (2010) define understanding as "a measure of the quality and quantity of connections that an idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding" (p. 24). Understood in this context as associations, or relationships between concepts, representations, and processes, the act of making connections is fundamental to learning mathematics (Boaler, 2002, Richland et al., 2012; Skemp, 1978) and to learning more generally (Caine & Caine, 1991). Connections have a physiological basis in the brain as neural synapses and pathways are formed to create neurological networks (della Chiasa, 2013).

Network

In the context of mathematics education, a network generally refers to an organised collection of nodes (vertices or points) that are connected together by arcs (lines or edges). Networks have been used to describe neurological synapses and pathways in the brain (della Chiesa, 2013) and as a metaphor to describe Big Ideas (Askew, 2013).

Process

Refers to a series of actions or steps to achieve a goal (e.g., the manufacturing process). In a mathematics education context, processes refer to a range of behaviours important in learning and using mathematical knowledge (e.g., analysing, comparing, discussing, explaining, generalising, measuring, problem solving, reasoning, synthesising).

Targeted teaching

Targeted teaching, as defined by Siemon (2006, 2011) is specifically concerned with Big Ideas. It is a form of differentiation that addresses students' specific learning needs in relation to a small number of Big Ideas in Number without which their progress in school mathematics will be seriously impacted.

Web

As in a spider's web – this term is generally used as a metaphor for the connections implicit within and between Big Ideas, for example, 'webs of meaning' (Thompson & Saldanha, 2003).

EVIDENCE BASE

Big Ideas and Understanding

To the extent that Big Ideas can be viewed as organised networks of connections and relationships arrived at as a result of mathematical activity, they have a critical role to play in learning mathematics with understanding. According to Hiebert and Carpenter (1992), understanding can be viewed as a "process of making connections, or building relationships, either between knowledge already internally represented or between existing networks and new information" (p. 80). The Australian Curriculum: Mathematics also views understanding as a process:

> ***Students build understanding when they connect related ideas, when they represent concepts in different ways, when they identify commonalities and differences between aspects of content, when they describe their thinking mathematically and when they interpret mathematical information[1].***

Making connections and building relationships is fundamental to the growth of mathematical understanding. This is an ongoing process that can proceed incrementally without us realising it or as the result of a sudden insight – the 'aha' moment when a relationship or connection suddenly comes to conscious attention.

1 https://australiancurriculum.edu.au/f-10-curriculum/mathematics/key-ideas/

WHAT ARE BIG IDEAS?

Many attempts have been made to characterise what constitutes a Big Idea for the purpose of developing a deep, well-connected understanding of mathematics (e.g., Askew, 2013; Charles, 2005; Kuntze, 2011; Schifter & Fosnot, 1993; Schreiber, 1983; Schweiger, 2006; Siemon, 2008)[2]. However, while these attempts have generated a large number of potential Big Ideas, they have not resulted in a shared sense of what qualifies as a Big Idea for this purpose.

The variation in the literature on Big Ideas have led some authors to conclude that we may never arrive at a shared understanding of what constitutes a Big Idea for the purposes of supporting a coherent approach to the teaching and learning of school mathematics (e.g., Charles, 2005; Clarke et al., 2012). However, a closer look is warranted as it may shed some light on which Big Ideas might be more useful than others in developing a deep understanding of mathematics over time. This discussion will focus on the two dimensions of variation: (i) *size* and (ii) *mode of expression*.

Size (i.e., how big is a Big Idea?). The content of school mathematics has always been subjected to categorisation at the macro level (e.g., number, measurement, chance). The categories vary over time in response to changes in the discipline (e.g., 'new' maths of the 1980's) and societal expectations (e.g., the PISA 2022 Mathematics Framework[3]). However, while these categories can be helpful in identifying fundamental mathematical practices (e.g., Bishop's *designing, locating,* and *playing*), and highlighting connections (e.g., Steen's *pattern),* they are too 'big' to inform teacher's everyday practice. A more refined set of key ideas and strategies and the links between them is needed to inform teaching and scaffold student learning (Askew, 2013; Charles, 2005; Siemon et al, 2012; Tout, & Spithill, 2015).

> ***Big Ideas need to be big enough that it is relatively easy to articulate several related ideas … Big Ideas need to be useful to teachers, curriculum developers, [and] test developers … If a Big Idea is too big … its usefulness for these audiences diminishes. (Charles, 2005, p. 11)***

Mode of expression. The issue here is that some Big Ideas are expressed as **concepts** (e.g*., function, infinity*) while others are expressed as **processes** (e.g., *explaining, structuring).* As Askew (2013) points out

> ***[this] is problematic in terms of teaching implications. In what sense are verbs such as 'combining' and 'locating' Big Ideas and how do they compare with mathematical objects such as 'algorithm'? How does one square off an idea such as 'invariance' with something equally broad but very natural such as 'playing'? (p. 6)***

The variation in expression appears to be related to *how Big Ideas are identified.* One approach, described by Hiebert and Carpenter (1992) as '**top-down**', is based on an analysis of mathematical structure. Or, as described by Charles (2005, p. 10) on a "careful analysis of mathematics concepts and skills … that looks for connections and commonalities that run across grades and topics". Big Ideas identified in this way tend to be expressed as *declarative statements*, for example, "fractions, decimals, and percentages all present different ways to represent a multiplicative relationship between two quantities" (Askew, 2013, p. 8).

2 A summary of these is included in Appendix 1.
3 https://pisa2022-maths.oecd.org

Another approach to identifying Big Ideas, which is similar to Hiebert and Carpenter's (1992) **'bottom-up'** approach, is based on exploring and documenting the increasingly sophisticated understandings constructed by students over time as they engage in mathematical activity (e.g., Confrey et al, 2014; Siemon et al, 2006, 2019; van den Heuvel-Panhuizen, 2016). The Big Ideas identified from this perspective tend to be expressed in terms of evidenced-based descriptions of key mathematical concepts and processes over time, for example, equipartitioning (Confrey et al., 2014) and multiplicative thinking (Siemon et al., 2006, 2012).

In my view both approaches have a role to play in helping to "represent mathematics as a coherent and connected enterprise" (NCTM, 2000, p. 17) – the first by identifying important learning goals that encompass powerful connections, the second by identifying what makes a difference to students' mathematics learning over time.

BIG IDEAS BASED ON AN ANALYSIS OF MATHEMATICAL STRUCTURE (TOP DOWN)

The most commonly cited view of a Big Idea from this perspective is the one proposed by Charles (2005)

> *Big Idea is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole (p. 10).*

Charles identifies twenty-one 'Big Ideas' in mathematics and provides examples of *mathematical understandings* for each (e.g., see Figure 1). While each of his Big Ideas is named, Charles makes it clear that it is the *statement,* not the name, that is the Big Idea.

In a similar vein, Askew (2013) offers five Big Ideas, four of which are included in Charles' list. Askew qualifies what he means as a Big Idea as being (1) culturally significant, (2) conceptually 'big,' and (3) pedagogically 'big'. For example, he makes the point that *place value* is a Big Idea because it is *culturally significant* (i.e., in the history of mathematics); *conceptually 'big'* in that it "helps learners come to understand mathematics as a network of interconnected ideas, not a series of separate ones"; and *pedagogically 'big'* in that later difficulties with decimals might be avoided if learners were to "meet the Big Idea of place value as a multiplicative scaling process based on powers of ten" (p. 8).

Figure 1: *Two examples of Big Ideas from Charles (2005)*

Big Idea #2

THE BASE TEN NUMERATION SYSTEM
The base ten numeration system is a scheme for recording numbers using digits 0-9, groups of ten, and place value.

Examples of Mathematical Understandings:

Whole Numbers

- Numbers can be represented using objects, words and symbols.
- For any number, the place of a digit tells how many ones, tens, hundreds, and so forth are represented by that digit.
- Each place value to the left of another is ten times greater than the one to the right (e.g., 100 = 10 x 10).
- You can add the value of the digits together to get the value of the number.
- Sets of ten, one hundred and so forth must perceived as single entities when interpreting numbers using place value (e.g., 1 hundred is one group, it is 10 tens or 100 ones).

Decimals

- Decimal place value is an extension of whole number place value
- The base-ten numeration system extends infinitely to very large and very small numbers (e.g., millions & millionths).

Big Idea #17

MEASUREMENT: Some attributes of objects are measurable and can be quantified using unit amounts.

Examples of Mathematical Understandings:

- Measurement involves a selected attribute of an object (length, area, mass, volume, capacity) and a comparison of the object being measured against a unit of the same attribute.
- The longer the unit of measure, the fewer units it takes to measure the object.
- The magnitude of the attribute to be measured and the accuracy needed determines the appropriate measurement unit.
- For a given perimeter there can be a shape with area close to zero. The maximum area for a given perimeter and a given number of sides is the regular polygon with that number of sides.

Note: From "Big Ideas and Understandings as the Foundation for Elementary and Middle School Mathematics", by R. Charles, 2005, *Journal of Mathematics Education Leadership,* 7(3), pp. 13-14. Copyright NCSM 2005. Reprinted with permission.

Clearly, Charles' list of Big Ideas and mathematical understandings and Askew's insights into Big Ideas provide a useful basis for organising what is taught into "coherent *networks of connected content structured around powerful ideas*" (Brophy, 2004, p. 256, emphasis in the original). However, while Tout and Spithill (2015) include some examples of how this might be achieved in discussing their 'clustered curriculum-based' approach to using Big Ideas, **neither the Big Ideas identified from this perspective, nor the curriculum-based examples, provide advice about what connections should be considered when, or which ones are more important than others.**

Moreover, based as they are on an analysis of mathematical structure, or, in Askew's case, what research and his own experience "suggest would improve learning" (p. 8), this approach will inevitably produce a very long list of Big Ideas as different people will have different views about what qualifies as a Big Idea for the purposes of teaching mathematics. However, this is not necessarily a problem because the **real benefit of considering Big Ideas from this perspective is that it engages teachers in a deeper discussion of the mathematics they are teaching making it more likely that they will look for connections and press for understanding** (Clarke et al., 2012).

BIG IDEAS BASED ON EVIDENCE OF STUDENT UNDERSTANDING OVER TIME (BOTTOM-UP)

Big Ideas identified from this perspective tend to be associated with evidenced-based descriptions of key mathematical **processes** over time (e.g., *unitising, relational thinking; equipartitioning*). Key **concepts** such as *place value*, *equivalence*, and *variation*, which are included in both Charles (2005) and Askew (2013), are also recognised as Big Ideas from this perspective. However, these tend to be described in terms of evidenced-based mathematical activity rather than as declarative statements (e.g., Day, et al., 2019; Fosnot & Jacob, 2013; Rogers, 2014; Siemon et al, 2012). While deeply connected to the structures of mathematics, the Big Ideas identified by the bottom-up approach are 'big' because they are critical to mathematics and because they are big leaps in the development of children's reasoning" (Fosnot & Dolk, 2001, p. 10).

Evidenced-based approaches to identifying the Big Ideas for the purpose of teaching and learning can be seen in the work on learning progressions/trajectories (e.g., Confrey et al., 2014, 2017; Clements & Sarama, 2009; Siemon et al., 2006, 2019).

An example of the Big Ideas from this approach can be seen in Confrey et al. (2017) whose nine Big Ideas (see Figure 2) frame a digital learning system (DLS) that provides users with the means "to navigate the content of middle school mathematics around big ideas and research-based learning trajectories" (p. 719). In this case, each of the Big Ideas is underpinned by evidenced-based 'relational learning clusters' that represent

> *a rich cognitive network of closely related ideas: a set of one or more constructs that should be learned in relation to each other … Constructs are arranged from bottom to top, illustrating their ordering in instruction; constructs at the same horizontal level can be introduced in either order or taught simultaneously. (p. 720)*

Figure 2: Four fields and nine Big Ideas (Confrey et al, 2017)

Algebra	Number	Statistics & Probability	Geometry & Measurement
			Measure, compose, and scale perimeter, area, and volume
Algebraically relate, express, modify, and evaluate unknown quantities	Position, compare, and operate on one dimensional quantities	Display data and use statistics to measure centre and variation in distributions	Compose, characterize, and transform lines, angles, and polygons
Represent and use relations and functions of two variables	Compare quantities to operate and compose with ratio, rate, and percent	Use probability to measure chance and model chance events to make informal inferences	Represent and explore Pythagorean Theorem and polygons using coordinate points

Note: From "Scaffolding learner-centred curriculum coherence using learning maps and diagnostic instruments designed around mathematics learning trajectories", by J. Confrey, G. Gianopolous, W. McGowan, M. Shah, & M. Belcher, 2017. *ZDM Mathematics Education* 49. P. 724. (www.https:doi.org/10.1007/s11858-017-0869-1). Reprinted with permission.

The advantage of framing Big Ideas in terms of evidenced-based descriptions of mathematical activity over time is that they can be observed and acted upon by teachers as they develop through different levels of complexity and as they are applied in different mathematical contexts. While this makes it difficult to completely define a Big Idea from this perspective to my mind those that are framed in this way are more likely to inform teacher's everyday decision making than Big Ideas expressed as declarative statements, particularly where the underpinning research provides diagnostic tools and teaching advice aimed at identifying and progressing student learning.

TO SUM UP – WHAT ARE BIG IDEAS IN SCHOOL MATHEMATICS?

It is unlikely that top-down analyses will lead to a definitive, agreed set of Big Ideas and advances in research will ensure that the Big Ideas identified from a bottom-up perspective will inevitably evolve over time. Nevertheless, I do think it is possible to come up with a with a common set of criteria (Figure 3) as a means of deciding which Big Ideas might be more useful than others for the purposes of identifying connections and supporting mathematics learning with understanding.

A later section will consider a particular subset of Big Ideas that our research and my reading have led me to regard as critical for students' mathematics learning but before this, a discussion of why Big Ideas are important in mathematics education.

Figure 3. Criteria for Big Ideas

Big Ideas in school mathematics

- are "mathematically big, conceptually big, and pedagogically big" (Askew, 2015, p. 13), that is, they have a basis in mathematics but also in how mathematics is taught and learnt (Fosnot & Dolk, 2001; Watson et al, 2013)
- involve semi-ordered structures of supporting, interrelated ideas and processes often described as maps, networks, or webs (Confrey et al., 2014; Hurst, 2019; Siemon, 2006; Van de Walle et al., 2010)
- grow in depth and complexity over time as they are connected with other aspects of mathematics and as they are applied in a broader range of contexts (Hiebert & Carpenter, 1992; Ma, 1999; Mason et al, 2009)
- provide a basis for a more coherent approach to the teaching and learning of mathematics that seeks to build connections, values multiple strategies and reasoning, and supports further learning with understanding (Confrey et al., 2017; Siemon, 2021; Siemon et al., 2012, 2019; Askew et al., 1997).

(Siemon, 2021)

The concept of Big Ideas is powerful because it assists teachers in developing a coherent overview of mathematics. But more importantly it enables students to develop a deeper understanding of mathematics and its interconnectedness, both within the world of mathematics and between the world of mathematics and the real world.

(Tout & Spithill, 2015, p. 19)

Learning with Understanding

It is widely recognised that learning with understanding is maximised where students have the opportunity to make connections within a specific area of mathematics, between different areas of mathematics, and between mathematics and their lived lives (Freudenthal, 1973; Hiebert & Carpenter, 1992; Tout & Spithill, 2015; van den Heuvel-Panhuizen, 2016; Watson et al., 2013). Students learn more and retain more of what they have learnt when it is connected to a network of related ideas and experiences (Caine & Caine, 1991; Charles, 2005; National Research Council, 2000). A focus on Big Ideas means focusing on connections, which once acquired increases the likelihood that the new learning will be applied to other situations.

> *A key finding in the learning and transfer literature is that organizing information into a conceptual framework allows for greater 'transfer'; that is, it allows the student to apply what was learned in new situations and to learn related information more quickly (National Research Council, 2000, p. 17)*

> *It has been known for many years that the understanding of big ideas leads to more flexible and generalizable knowledge use, improves problem solving, makes it easier to make sense of and master new facts and procedures, and enables transfer (Niemi et al., 2006, p. 3)*

Curriculum Coherence

One of the most damaging aspects of school mathematics, particularly as it is represented in commercial texts and computer-based mathematics programs is the atomisation of the curriculum into a seemingly endless list of discrete 'topics' where 'coverage' becomes the major goal of planning, the predominant mode of learning is 'example/demonstrate-practice-practice', and the majority of those exercises remain as they were observed by Vincent and Stacey in 2008 as relatively low-level, skill based repetitious exercises. This is hardly conducive to providing students with the opportunity to engage in sustained problem solving, extended investigations, or robust discussion. **Adopting a Big Ideas approach to curriculum planning helps ensure that those areas of mathematics that are more critical for student learning are given the attention they deserve.** That is, Big Ideas can be used as a filter to make decisions about what mathematics to consider, when, and how to maximise students opportunity to learn mathematics with understanding.

Key to effective teaching

A focus on Big Ideas is a focus on making connections. Research on the characteristics of effective teachers (e.g., Askew et al, 1997; Clarke & Clarke, 2002; Ma, 1999, 2010) has consistently found that the most effective teachers of mathematics maintain an unrelenting focus on learning with understanding. As a result, they are aware of and actively seek out opportunities for students to make connections, notice similarities and differences, generate and test conjectures, and look for generalisations. One of the best-known studies in this area is the one by Askew et al (1997) that explored what made some primary teachers more effective in improving student outcomes than others. While the research produced quite a long list of important qualities, reporting on the findings of this study some years later Askew (2012) noted the following.

> *The key thing that distinguished effective teachers from the others was … having a connectionist orientation to teaching and learning, in particular:*
>
> - *making connections within mathematics, both between different aspects of mathematics, for example, addition and subtraction or fractions, decimals, and percentages, and between different representations of mathematics – symbols, words, diagrams and objects;*
> - *making connections with children's methods – valuing these and being interested in children's thinking but also sharing other methods. (p. 35)*

Impact student outcomes

Targeted teaching using evidenced-based formative assessment materials linked to Big Ideas that make a difference (refer to next section on p. 10) has been found to make a significant difference to student learning outcomes where implemented flexibly within a well-structured, pedagogically-sound mathematics program (e.g., Breed, 2011; Goss et al. 2015; Siemon et al., 2018).

There is no one way to implement a targeted teaching approach – some schools have set aside one lesson a week or a double period a fortnight to enable students to work on activities designed to progress their learning in relation to the Big Ideas (e.g., see Case Studies on the *Growing Mathematically – Multiplicative Thinking website*[4]). Note: Targeted teaching is not ability grouping by another name – the evidence for mixed ability teaching in mathematics is overwhelming (e.g., Boaler & Foster, 2021). All students need the opportunity to learn alongside their peers in the context of challenging but accessible and purposeful tasks (e.g., Clarke & Roche, 2018; Sullivan, 2011). See Ability Grouping monograph (Siemon, 2022) for more on this topic.

4 http://www.mathseducation.org.au/online-resources/growing-mathematically/context/

BIG IDEAS THAT MAKE A DIFFERENCE

As indicated earlier, the following section discusses a subset of Big Ideas that I regard as critical for students' mathematics learning. It starts with a consideration of the evidence.

The *Middle Years Numeracy Research Project* (Siemon & Virgona, 2001) used rich assessment tasks to explore Year 5 to 9 students' sense of number, space, measurement, and data. The project found a five to seven-year range in mathematics achievement in Years 5 to 9 that was almost entirely due to the extent to which students had access to *multiplicative thinking*.

The *Scaffolding Numeracy in the Middle Years* (SNMY) project also used rich assessment tasks to investigate the development of multiplicative thinking in Years 4 to 8 (Siemon et al., 2006). This confirmed the results of the previous project and produced an evidenced-based learning progression for multiplicative thinking, validated assessment options, and teaching advice in the form of the *Learning Assessment Framework for Multiplicative Thinking* (LAF, see Appendix 2). Importantly, analysis of the learning progression data for the purpose of preparing the teaching advice identified **a small number of underpinning Big Ideas in Number that need to be in place by key stages in schooling to ensure students have access to multiplicative thinking as it develops over time.**

These underpinning Big Ideas were used to structure the Assessment for Common Misunderstanding materials (AfCM) which were published on the Victorian Department of Education website in 2006[5]. The AfCM materials were based on the performance-based tasks and teaching advice developed to support pre-service education and identify starting points for teaching in remote Indigenous schools in the Northern Territory (e.g., Siemon et al., 2004). The subsequent use of both the SNMY and the AfCM materials to identify and address student learning needs in relation to these Big Ideas convinced me that they were critical in ensuring all students have the opportunity to productively participate in school mathematics and progress their learning.

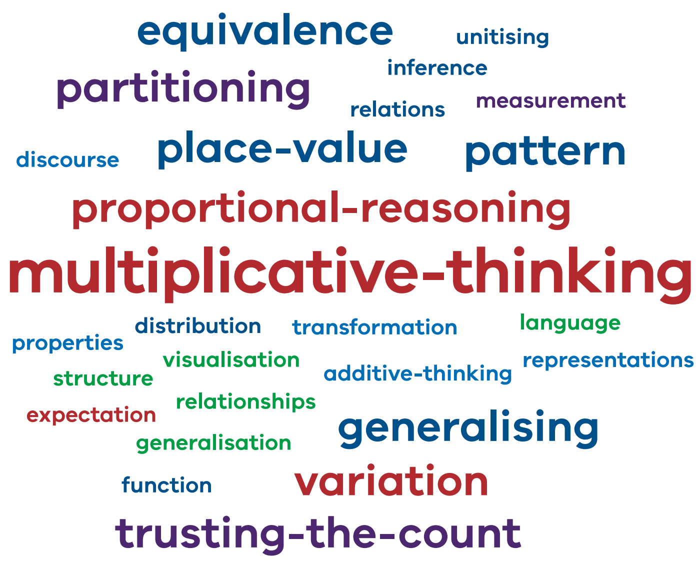
A Big Idea from my perspective - As a result, a Big Idea from my perspective and for this purpose is:

- an idea, strategy, or way of thinking about some key aspect of mathematics **without which, students' progress in mathematics will be seriously impacted**,
- **encompasses and connects many other ideas and strategies**,
- serves as an *idealised cognitive model* (Lakoff, 1987), that is, it **provides an organising structure or a frame of reference** that supports further learning and generalizations,
- cannot be clearly defined but **can be observed in activity** (Siemon, 2008; Siemon et al. 2012).

More recently, the *Reframing Mathematical Futures* Project investigated the development of algebraic, geometrical, and statistical reasoning in Years 5 to 10 to produce evidenced-based learning progressions, formative assessment options, and teaching advice framed in terms of the respective Big Ideas identified for each area of mathematical reasoning (e.g., Siemon et al, 2018b, 2019). Additionally, this project found a strong relationship between mathematical reasoning and multiplicative thinking and confirmed that the latter was responsible for the seven-year range in mathematics achievement in the middle years (Siemon, 2016, Siemon et al, 2018a).

The Big Ideas identified collectively by this work are shown in Figure 4. While it is not possible to consider all of these, a number of the key ones are described below, starting with a brief summary of those included in the AfCM. More detailed information can be found in the AfCM materials on the DET website, in Siemon et al. (2021), and in the teaching advice associated with the resources listed under the Big Ideas Audit below.

Figure 4: A word cloud of evidenced-based Big Ideas



Note: Based on the Assessment for Common Misunderstandings, Scaffolding Numeracy in the Middle Years, Growing Mathematically, Reframing Mathematical Futures II Projects (Siemon, 2006; Siemon et al., 2012; Siemon et al., 2019)

5 A table showing the relationship between the SNMY teaching advice and the AfCM Big Ideas is included in Appendix 2.

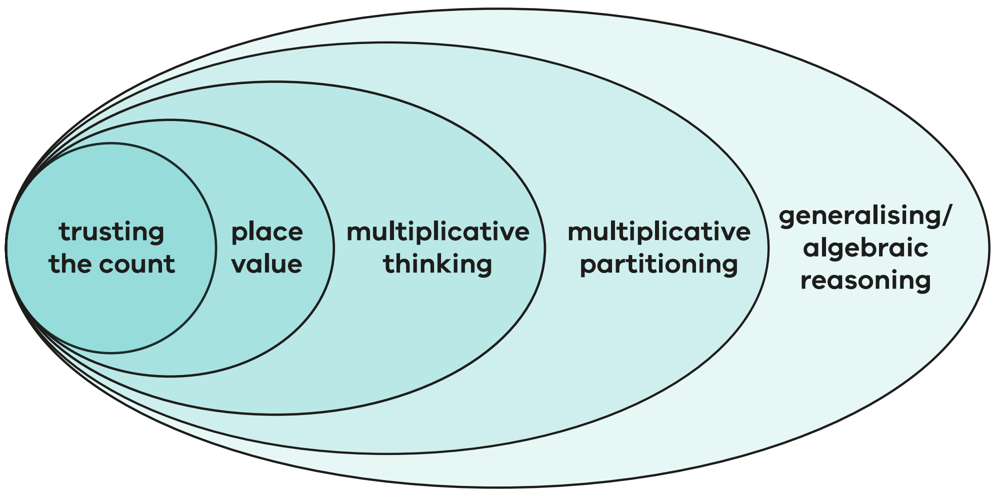
Table 1. The AfCM underpinning Big Ideas for multiplicative thinking by key stages of schooling

By end of	Big Idea	Indicated by
Prep/K	Trusting the Count	Access to flexible mental objects for the numbers to ten based on part-part-whole knowledge derived from subitising and counting
Year 2	Place-value	Capacity to recognise and work with place-value units and view larger numbers as counts of these units rather than collections of ones. Appreciates structure in terms of '10 of these is 1 of those'
Year 4	Multiplicative Thinking	Initial Ideas – Works with multiple representations of multiplication and division (e.g., the 'for each', 'times as many' and 'area' ideas). Moving to factor-factor-product idea, efficient strategies for multiplication facts
Year 6	Equi-partitioning	Uses partitioning strategies to construct line and areas models for fractions and decimals, uses representations to compare, order and locate fractions and decimals on number lines, recognise that numbers can be divided to create new numbers, solves simple problems involving fractions
Year 8	Proportional Reasoning	Ability to recognise and work with an extended range of concepts for multiplication and division including rate, ratio, percent, solves problems involving intensive quantities and proportional relationships
Year 10	Generalising	Capacity to recognise and represent patterns and relationships in multiple ways including symbolic expressions, devise and apply general rules and properties

Note: From the Assessment for Common Misunderstandings (Siemon, 2006; Siemon et al, 2012)

It is important to note that the 'By the end of' year levels are not when we start to teach these core ideas, it is when they need to be in place for students to have a realistic chance of successfully engaging with the mathematics that follows to build a deep understanding of multiplicative thinking. **Every teacher at every year level has some responsibility for the development of each of these Big Ideas** (e.g., Siemon et al, 2021) and since the AfCM was published, Hurst and Hurrell (2014) have come up with a really powerful way of illustrating this point (see Figure 5).

Figure 5: An alternative representation of the AfCM Big Ideas (Hurst & Hurrell, 2014)



Note: From "Developing the Big Ideas", by C. Hurst & D. Hurrell, 2014. International Journal of Educational Studies in Mathematics Education, 1(2). p. 4 (www.https:doi.org/10.1007/s11858-017-0869-1). Reprinted with permission.

The following discussion of some of the evidenced-based Big Ideas in Figure 4 is included to illustrate the critical role of Big Ideas in supporting connections within and between other Big Ideas and other aspects of mathematics. The problems can be explored and discussed in teaching teams.

Unitising – This really big idea has its origins in the recognition of small collections as composite units (e.g., 7 ones is 1 seven). Initially, this idea is arrived at through visual recognition (i.e., subitising), for example, 6 is recognised on a dot dice as 'a six' and a bundle or stack of 10 ones is seen as 1 ten. Eventually, students need to recognise that any number or quantity can be conceived of as a unit.

> *Unitising is a cognitive process that involves assigning a 'chunk' of a given quantity as a unit to facilitate thinking about and working with that quantity. Unitising builds on the idea of units of measurement but it is broader than that as the units are determined by the solver in relation to the particular situation/context. (Siemon et al., 2021, p. 591)*

This idea is critical to building an understanding of place-value, but ultimately the capacity to unitise underpins *multiplicative thinking, partitioning, and proportional reasoning* (e.g., Lamon, 1999; Siemon et al., 2021). For example, consider what is involved in solving the following problem.

Problem

Samantha's Snail travelled 1.59 metres in 6 minutes. Jeanie's Snail took 4 minutes and 30 seconds to travel 126 cm. How long did each snail travel in 9 minutes?

From AfCM Level 5 Tools (Siemon, 2006)

Part-Part-Whole – This Big Idea could be described as a *mathematical structure* in the sense that it underpins all additive situations. Initially this involves the understanding that collections to ten can be conceived of in terms of their parts (e.g., 8 is 1 more than 7, 2 and 6, 5 and 3) or as they are a part of a larger collection (e.g., 8 is 1 less than 9, or 2 less than 10). Generally referred to as 'part-part-whole knowledge' (Siemon et al., 2021, Van De Walle, 2010), this is a key component of *trusting the count, place value, and additive thinking*, for example,

- part-part-whole knowledge based on subitising and visual imagery is key to establishing the numbers to ten as mental objects which is a core component of trusting the count (Siemon, 2006; Siemon et al., 2021)
- recognising numbers in terms of their parts is necessary for the development of the 'make-to -ten' mental strategy for addition (e.g., understanding 6 as 4 and 2 and 8 as 2 less than 10 supports the thinking: 8 ... 10 ... 14)
- renaming numbers in terms of their parts or as they are a part of a larger number (i.e., representing numbers in an equivalent form) supports more efficient calculation (e.g., 173 – 47 = 173 - 50 + 3 = 126)
- renaming numbers in terms of their place-value parts (e.g., 4003 is 400 tens and 3 ones), which is one of the strongest indicators of deep place value knowledge (Rogers, 2014), can also be used to support efficient calculation
- in more complex additive problems, 'is it a part or a whole?' is a useful problem-solving strategy
- while not to be confused with the part-whole fraction idea, the part-part-whole structure can be useful in thinking about complements (e.g., that 5 eighths is 3 eighths less than 1).

To my mind *additive thinking* is not a Big Idea in its own right – it is a direct consequence of *trusting the count* and *place value* and it is strongly associated with *relational thinking* (see below). That said, distinguishing between parts and wholes is not always easy as the following problem demonstrates.

Problem

Sam and Joseph each had a shorter sister, and they argued about who was more taller than his sister. Sam won the argument by 14 centimetres. He was 186 cm tall; his sister was 87 cm; and Joseph was 193 cm tall. How tall was Joseph's sister?

(Smith & Thompson, 2007, p. 32)

Place Value – Understanding the structure of the base 10 system of numeration requires the ability to "mentally move flexibly within and between different unit systems – such as hundred, tens and ones" (Hunting, 2003, p. 218). This is a multiplicative process. The digits 0 to 9 are used to indicate *how many* and place to indicate *how much*, where each place represents a different power of ten. Appreciating place value requires the recognition that what is being counted in any one place is 10 times greater (or 10 times less) than what is being counted in the adjacent place. That is, that 'ten of these is 1 of those' and, conversely, '1 tenth of these is one of those'. This Big Idea is difficult to teach and learn as illustrated in the problems below (see also Tool 4.6 from AfCM). In Years 1 to 8 it needs to be considered on a regular basis throughout the year in multiple contexts. There are a number of very well-known misconceptions associated with decimal place value (e.g., see Siemon et al., 2021; Steinle, 2004) many of which teachers have inadvertently contributed to (e.g., when multiplying or dividing by powers of ten the decimal point 'moves' when it is the numbers that move, not the decimal point).

Problem

(i) Which is the longest distance: 0.1405 km, 145 m, 1450 cm, or 14,050 mm?

(ii) Multi-base arithmetic blocks (MAB) are used to model numbers in base ten, what would the blocks look like in base 2?
How would 503 be written in base 2?

(iii) Write 2023 in one or more historical number systems (e.g., Roman, Egyptian, Chinese, Greek, Babylonian, Mayan)?

(Siemon et al., 2021)

Equivalence – This is a really Big Idea in mathematics that underpins all forms of computational and algebraic thinking. While it can be described in relatively simple terms, for example, "There are infinitely many ways to represent numbers, measures, and number sentences" (Askew, 2013, p. 9), equivalence is often misunderstood (Warren & Miller, 2021). For instance, a major problem arises in the early years when children are introduced to the equal sign in the context of simple addition and subtraction problems. While it very easy to say, "3 oranges and 4 more oranges 'makes' 7 oranges", where this is represented as 3 + 4 = 7, many children assume that the equal sign means 'makes' or 'the answer comes next' (Fosnot & Jacob, 2010; Warren & Miller, 2021). This and the inability to regard an expression such as m + 4 or 3n as objects (i.e., as numbers that can be operated on as opposed to processes to be carried out) have been identified as a major reason why students have difficulty with algebra (e.g., Kieran, 1992; National Research Council, 2000).

Problem

Record as many different equivalent representations of 64 as you can in 2 minutes. Then share and discuss with colleagues. How could you prove that 2(n-1) = 2n – 2?

(Fosnot & Jacob, 2010)

Relational Thinking – According to Stephens (2006), "*relational* thinking depends on children being able to see and use possibilities of *variation* between numbers in a number sentence" (p. 479), for example, be able to explain without calculating why 54 + 39 = 53 + 40 and find the missing number in 53 – 27 = ☐ - 30. Relational thinking involves working with *equivalence*, *part-part-whole* knowledge, and recognising *properties* such as commutativity, distributivity, and associativity as well as additive and multiplicative inverses, all of which contribute to *structural thinking* more generally (Mason et al., 2009).

Structural Thinking – In mathematics education, *structure* has been used to refer to the appreciation and use of *patterns*, arrays, and *multiple representations* (e.g., Mulligan & Mitchelmore, 2009; Mulligan et al., 2009), the base ten system of numeration (e.g., Thomas et al., 2002; Young-Loveridge & Bicknell, 2016), *mathematical modelling* (e.g., English, 2003), *algebraic thinking* (e.g., Blanton et al, 2019), *geometrical* reasoning (e.g., Seah & Horne, 2019), and *statistical reasoning* (e.g., Callingham, et al., 2019). From a mathematics discipline perspective, *mathematical structure* has been described as the "identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set" (Mason et al., 2009, p. 10). However, 'seeing' a pattern or relationship in a particular context is not sufficient. An appreciation of mathematical structure requires that the inherent properties, pattern or relationship is recognised more generally in different contexts (Mason et al., 2009; Vale et al., 2011; Warren & Miller, 2013). An example of this is evident in the following problem from Mason et al. (2009, p. 24).

Problem

Think about the following mathematical sentence:

$$18 + \square = 20 + \square$$

Box A Box B

(a) Can you put numbers in Box A and Box B to make three correct sentences like the one above?

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

(c) If instead of 18 and 20, the first number was 226 and the second number was 231 what would be the relationship between the numbers in Box A and Box B?

(d) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

Multiplicative Thinking – The capacity to reason with relationships between different quantities in ways that go beyond repeated addition is needed to develop a deep, well-connected understanding of rational number and *proportional reasoning* as it occurs in a variety of contexts. Multiplicative thinking is also related to *measuring, equipartitioning* and the *structure* of the base ten system of numeration. Although young children understand the notion of a many-to-one count (a ratio idea), multiplication is typically introduced as a count of equal groups. While repeated addition has its place, students need to understand multiplication in term of the 'times as many' and 'for each' ideas for multiplication by the end of Year 4 if they are to engage productively and meaningfully with fractions, decimals, percent, rate, ratio, and proportion in later years. Recent research has also found a strong correlation between multiplicative thinking and *algebraic, statistical,* and *geometrical reasoning* (Siemon et al., 2019). The shift from additive to multiplicative thinking is not easy and may take considerable time to achieve as it requires a "cognitive reorganisation on the part of learners" (Fosnot & Jacob, 2010, p. 14). The following quote from Thompson and Saldanha (2003) is a good example of a powerful connection

> *When students understand the numerical equivalence of measuring and partitioning they understand that any measure of a quantity induces a partition of it and that any partition of a quantity induces a measure of it. (p. 30)*

CHALLENGES IN TEACHING WITH BIG IDEAS

TEACHER KNOWLEDGE AND CONFIDENCE

Adopting of a Big Ideas approach to teaching and learning mathematics requires confident knowledgeable teachers (Askew, 2013; Ball & Bass, 2009; For instance, Clarke et al. (2012) found that many primary teachers experience difficulty identifying and/or expressing 'the most important idea' they would focus on in teaching the next topic. A similar result was found for German pre-service teachers by Kuntze et al. (2011) who reported that many "were unable to discern big ideas behind mathematical content and to link elements of content matter according to these big ideas" (p. 2717).

> ***Because Big Ideas have connections to many other ideas, understanding Big Ideas develops a deep understanding of mathematics. When one understands Big Ideas, mathematics is no longer seen as a set of disconnected concepts, skills, and facts. Rather, mathematics becomes a coherent set of ideas. (Charles, 2005, p. 10)***

By far the most recognised study in this area is Ma's (1999) comparison of U.S. and Chinese primary teachers' mathematics knowledge for teaching in relation to four topics: subtraction with regrouping, multi-digit multiplication, division with fractions, and calculating perimeter and area. While the U.S. teachers saw these as unrelated topics, tended to focus on procedures, and found the last two topics challenging, the Chinese teachers demonstrated deep, well-connected conceptual knowledge not only of each topic but also of the connections between them. Ma described this in terms of a *Profound Understanding of Fundamental Mathematics* (PUFM) by which she means "an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough" (p. 120).

This issue is particularly the case for out-of-field teachers and primary teachers whose opportunity to learn mathematics in the past may have been limited. However, professional learning focussed on the Big Ideas has been found to make a difference (e.g., Cooper et al., 2016; Fosnot & Jacob, 2010; Siemon et al., 2012; Vale et al., 2010).

DEPTH OVER BREADTH

One of the most persistent issues in mathematics teaching is deciding what to cover, when, and to what depth. This is an age-old problem that as Brophy (2004) notes is unlikely to ever be resolved.

> ***This tension between breadth of coverage and depth of topic development is an enduring dilemma that teachers have to manage as best they can; it is not a problem that you can solve in any permanent or completely satisfactory manner. … Critics routinely complain that textbooks offer "mile-wide but inch-deep" curricula featuring parades of disconnected facts instead of coherent networks of connected content structured around powerful ideas. (p. 34)***

BUILDING CONSENSUS

Even though it is highly unlikely that everyone will agree on what constitutes a Big Idea or what the Big Ideas are for the purposes of teaching and learning school mathematics (Charles, 2005), as Clarke et al. (2012) have pointed out the discussions are worth having particularly if it leads to a commitment to engage in further professional learning and/or a search for resources that represent a Big Ideas approach. Resources that support a Big Ideas approach will typically emphasise meaning-making over procedures, and connections over isolated topics. They will provide opportunities for students to explain and justify their thinking, compare and discuss solution strategies with peers, notice similarities and differences between problems; explore multiple representations; construct and test generalisations; and focus on important mathematical ideas and processes in the context of challenging but accessible and meaningful tasks (Hiebert & Carpenter, 1992; Sullivan, 2011; Watson et al., 2013). To this end, it is important that professional learning communities make it their business to consider not only how the Big Ideas described above might impact their decision-making about what should be taught, when, and how often but also that they seriously review the resources used.

ACCOMMODATING DIVERSITY

There are many reasons why not all students learn the same thing at the same time but grouping students by ability is not the solution (see Ability Grouping monograph) nor is it necessary to differentiate everything. Identifying where students are in relation to a small number of really Big Ideas and targeting those needs in rich, mixed ability settings has been found to make a significant difference to student engagement, participation, and learning outcomes (Siemon et al, 2006, 2012, 2018a). Resources to support the use of Big Ideas in this way are included below.

PLANNING

Adopting a Big Ideas approach to the teaching and learning of mathematics needs to be supported by school leadership strategically (i.e., in policy documents) and through the allocation of resources (e.g., qualified staff, professional learning, quality teaching and learning resources). As Clarke et al. (2012) have shown many teachers find it difficult to identify the most important ideas in a topic prior to teaching it – they suggest and I agree that in addition to professional learning opportunities that explore the Big Ideas, it is worthwhile engaging in team-based activities to discuss what they think the key ideas are and why. The concept mapping exercise suggested in the quote below is one way of doing this.

Teaching with the Big Ideas in Mathematics

SUPPLEMENTARY MATERIALS

Activities and reference material

WHAT DOES A FOCUS ON BIG IDEAS LOOK LIKE IN PRACTICE?

FIRST STEPS: WHAT CAN SCHOOLS DO TO ADOPT A BIG IDEAS APPROACH TO MATHEMATICS TEACHING?

Big idea teachers are not limited in their thinking by curriculum boundaries. Most importantly, big idea thinking encourages teachers to deconstruct and reconstruct their knowledge. Teachers can actively engage in this by beginning with a mathematical idea such as place value and building a concept map showing the various pieces of 'micro-content' that contribute to the development of the concept and considering how the content is connected.

(Hurst, 2019, p. 71)

Build professional capacity

The following activities should be considered at a whole staff meeting in primary schools, at a Faculty/Department meeting in secondary schools, or, given the mathematical nature and significance of Big Ideas, at a joint meeting of staff from both primary and secondary schools.

(i) Consider the word cloud shown in Figures 4 (slide 1 of the Big Ideas Provocation). What do you see, think, and wonder about in relation to your current approach to planning and teaching mathematics?

(ii) List the ideas that you think are the most useful in terms of helping you think about *connections (see key terms)* and discuss with colleagues.

(iii) To what extent are Big Ideas evident in your planning and in the resources you use to support the teaching and learning of mathematics? Consider undertaking a review of resources (see Resource Review below).

(iv) For a slightly different view on Big Ideas, download, read, and discuss 'Big Ideas in Mathematics Teaching' by Dave Tout and Jim Spithill (2015). Consider undertaking the activity on p. 12.

(iv) Primary teachers - Download and watch the short video, 'We've got to put some lines in the sand', which briefly discusses the importance of the Big Ideas of trusting the count, *place value* and *multiplicative thinking*. Do you agree/disagree on the importance of these Big Ideas? What do you know about your students understanding of these ideas? Consider using the Assessment for Common Misunderstanding materials on the DET website (see Big Ideas Audit below) to find out more.

(vi) To better understand the role of evidenced-based learning progressions and Big Ideas download, read, and discuss the following article from Teacher Magazine.

Resource Review

Review resources used to inform or support the teaching and learning of mathematics at the school (including current text and/or digital resources, planning documents and activities) with a view to determining how the mathematics to be learnt is presented to students. This is likely to require a whole day or at least a half-day meeting attended by all teachers of mathematics.

One way of doing this is to look at the development of a key area of mathematics such as place value or rational number knowledge over time (i.e., fractions, decimals, ratio, rate). Consider the following:

- ***connectedness*** to underpinning/overarching ideas, language, other aspects of mathematics – rank on a scale of 1 to 10, where 1 means little/no evidence and 10 means strong evidence
- ***predominant focus*** – assign a percentage to the relative amount of time and space given to each proficiency, that is, procedural fluency (skill development and practice), conceptual understanding, (explanations, discussion, connections), problem solving (challenging problems, tasks, strategies, opportunities to pose problems), and reasoning (justification, proof, logical arguments)
- ***representations*** – are multiple representations offered, are different solution strategies explored, what opportunities are there for discussion, comparing and contrasting solutions?
- ***organisation of content*** – is there any logic to the sequence of topics, how are exercises grouped (e.g., method of solution, type of numbers involved)?

Another way of doing this is to examine the time allocated to the different aspects of mathematics in scope and sequence charts or year or term planners. In this case, consider:

- ***time allocation*** – is this based on ensuring exposure (i.e., coverage), or are some areas considered in more depth or more frequently than others – if so, why? What proportion of the time is allocated to challenging but accessible tasks/problems as opposed to exercises?
- ***content decisions*** – are these based on the applicable year level curriculum content descriptors or something else?
- ***vertical progression*** – to what extent is content presented in one year level duplicated in the next, are key ideas reconsidered?
- ***horizontal connections*** – to what extent is content revisited in other topic areas, related to other aspects of mathematics at the same year level?

Secondary schools may want to use the PISA 2022 Mathematical Literacy Framework as a lens to consider how mathematics teaching is organised and resourced.

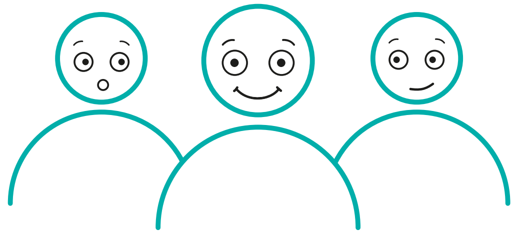
Big Ideas Audit

Use local data and/or known gaps in understanding to select and use one or more of the following resources to ascertain where students are at in relation to the Big Ideas discussed above. Discuss results in a team meeting and how the associated teaching advice might be used to address specific learning needs through targeted teaching in the context of mixed ability classrooms (see the monograph on Ability Grouping).

Table 2: Big Ideas Formative Assessment Options

Assessment for Common Misunderstandings (AfCM) Years F to 10	Short task-based individual interviews and targeted teaching advice Big Ideas: *Trusting the Count, Place Value, Multiplicative Thinking (initial ideas), Partitioning, Proportional Reasoning,* and *Generalising*
Scaffolding Numeracy in the Middle Years (SNMY) Years 4 to 10	Class-based assessment options and teaching advice aligned to an evidenced-based learning progression Big Idea: *Multiplicative Thinking*
Growing Mathematically – Multiplicative Thinking (GM-MT) Years 5 to 10	Additional SNMY options and teaching advice aligned to revised evidenced-based learning progression (includes related RMFII items) Big Idea: *Multiplicative Thinking*
Reframing Mathematical Futures II (RMFII) Years 5 to 10	Class-based assessment options and teaching advice aligned to evidenced-based learning progressions for algebraic, geometrical and statistical reasoning Big Ideas: *Algebraic Reasoning* (Pattern & Function, Equivalence, Generalisation), *Geometrical Reasoning* (Hierarchy & Properties, Transformations & Relationships, Geometric Measurement), *Statistical Reasoning* (Variation with Expectation & Randomness, Variation with Distribution & Expectation, Variation with Informal Inference)

ENGAGEMENT ACTIVITIES TEAM-BASED



1. The Importance of Making Connections

Work in teaching teams to complete and discuss slides 1 to 6 from the Big Ideas Provocation PowerPoint.

1. **Read:** 'Meaning versus Understanding' (slide 1) for not more than 2 minutes.

   Discuss: The text is understandable in that each sentence is well-constructed, and the meaning of each word is clear but what does it mean? What procedure is being referred to in the text? What are you thinking about as you try to make sense of this?

2. **Read:** 'The Power of Meaning (slide 2) for not more than 1 minute and try to commit the text to memory with as little effort as possible. After 1 minute, move to slide 3 and try to answer the questions. Discuss in terms of strategies used and degree of difficulty.

   Next read slide 4 for not more than 1 minute and try to commit the text to memory with as little effort as possible. After 1 minute, return to Slide 3 and try to answer the questions. Discuss in terms of strategies used and degree of difficulty. What made the second set of sentences easier to remember? Why?

   Next show slide 5 and try to commit the associations to memory for no more than 1 minute. Now try again using slide 6. What do you notice? Which was easier? Why?

   Reflect on these experiences and discuss in terms of the importance of making connections to prior knowledge and experience and the role of spatial structures in memory.

3. 'This Goes With That' (slide 7, adapted from Maths300) is a good example of a mathematics activity that has many connections within and between Big Ideas.

   Workshop this task in a team meeting having collected some appropriate data (e.g., number of cars by colour in the carpark). You will need some paper tape (25 mm paper striping is ideal for this purpose), a 100-bead string that can be tied at each end, and a 1 metre ruler.

   Next, discuss connections to Big Ideas and how you might use/adapt this task for the classroom. Note: Maths300 is a subscription-based resource.

2. Explore Statistical Thinking

(see Slide 8)

Access the Top-Drawer resources for Statistics. After an initial discussion of the Big Ideas of Statistics and Probability in teaching teams, individual team members could trial a year-level appropriate activity then report back their insights with the team.

3. Explore Relational Thinking

(see Slide 9)

Consider and discuss the problems on the Big Ideas Provocation power-point (slides 9 and 10) in teaching teams. Next, plan and trial a lesson that includes one or more relational thinking items appropriate to the year level, possibly as a Number Talk.

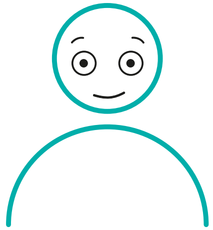
Now devise and offer three other problems – one that is of a similar level of difficulty (Consolidate), one that is a little bit more difficult (Stretch) and one that is more difficult again (Challenge). Students choose one and work in pairs to solve and prepare an explanation. Leave at least 10-15 minutes to discuss solution strategies at the end of the lesson. Note students' responses and share with colleagues to consider 'where to from here?'

4. Curriculum planning exercise

(see The STEM Agenda Monograph)

This approach to mathematics curriculum planning is designed to ensure that **priority is given to important mathematical ideas and the proficiencies** by generating time and space in the 'crowded curriculum' for integrated activities that are explicitly based on the mathematics. For advice on developing an integrated unit of work read and discuss Mike Askew's (2017) article *which was published in Common Denominator.*

ENGAGEMENT ACTIVITIES INDIVIDUAL



1. Personal reflection

How well do you know mathematics for teaching?

Download and print one of the SNMY, GM-MT, or RMFII formative assessment options in Table 2 above and without looking at the scoring rubrics or using a calculator complete as much as you can. Mark your work using the scoring rubric then read the relevant teaching advice. Are there areas of mathematics you need to work on to become more confident or to better understand the connections and the mathematics at the horizon? If so, two texts that you may find useful are Siemon et al. (2021) and Watson et al. (2013).

To read more, access the article by Ball and Bass (2009) it provides some interesting insights into the mathematics needed for teaching.

2. Explore your own knowledge of the Big Ideas

Work on your own or with a colleague to explore your knowledge of the Big Ideas using slides 11 to 16 from the Big Ideas Provocation Powerpoint:

- Slide 11 deals with place value
- Slides 12-14 consider multiplicative thinking
- Slides 15-18 explore partitioning
- Slides 19-21 involve proportional reasoning.

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APPENDIX 1:
Definitions of Big Ideas in Mathematics Education

Schreiber (1983)* *Universal & Central Ideas* *Cited in Kuntze et al. (2011)	*Universal Ideas* have "width (logical generality), richness (broad applicability and relevance in different areas of mathematics, and sense (anchoring within the realm of common sense, relevance and meaning in real life situations" (p. 69). *Central Ideas* are less general embodiments of universal ideas specific to particular areas of mathematics.
Schifter & Fosnot (1993) *Big Ideas*	Big Ideas are "the central organising ideas of mathematics – principles that define mathematical order" (p. 35).
Fosnot & Dolk (2001) *Big Ideas*	As per Schifter and Fosnot (1993) but add: "As such, they are deeply connected to the structures of mathematics. They are, however, also characteristic of shifts in learners' reasoning – shifts in perspective, in logic, in the mathematical relationships they set up … These ideas are 'big' because they are critical to mathematics and because they are big leaps in the development of children's reasoning" (p. 10).
Charles (2005) *Big Ideas & Understandings*	A *Big Idea* is "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p. 10). A *mathematical understanding* is an important idea students need to learn because it contributes to understanding the Big Idea.
Fosnot & Perry (2005) *Big Ideas*	*Big Ideas* are "learner-constructed, central organizing principles that can be generalized across experiences, and which often require the undoing or reorganizing of earlier conceptions. This process continues throughout development". (n.p.)
Schweiger (2006) *Fundamental Ideas*	*Fundamental Ideas* are those which "recur in the historical development of mathematics (time dimension), recur in different areas of mathematics (horizontal dimension), recur at different levels (vertical dimension), are anchored in everyday activities (human dimension)" (p. 68).
Siemon (2008) Big Ideas	A *Big Idea* is "an idea, strategy, or way of thinking about some key aspect of mathematics: without which, students' progress in mathematics will be seriously impacted; that encompasses and connects many other ideas and strategies; serves as an idealised cognitive model (Lakoff, 1987), that is, it provides an organising structure or a frame of reference that supports further learning and generalisations; cannot be clearly defined but can be observed in activity" (s. 10).
Kuntze et al. (2011) *Big Ideas*	*Big Ideas* are characterised as "having high potential for developing conceptual knowledge; having high relevance for building knowledge about mathematics as a science; supporting communication and mathematics-related arguments; encouraging reflection processes of teachers" (p. 2718).
Askew (2013) *Big Ideas*	A *Big Idea* "should be both culturally, that is mathematically, significant as well as individually and conceptually significant … However, I also suggest that, in putting together a collection of Big Ideas, there are pragmatic considerations that address pedagogical issues over and above theoretical ones" (p. 7).
Watson et al. (2013) *Key Ideas*	*Key Ideas* found by "identifying threads in the curriculum; identifying mathematical concepts that permeate mathematics; identifying mathematical concepts that seem to present students with difficulties; and identifying mathematical concepts that have strong implications for employment and citizenship" (p. 5).

APPENDIX 2:
Relationship between SNMY LAF and AfCM (to Year 8/9)

Year Level	SNMY Zone	Summary of SNMY Teaching Advice	AfCM Big Ideas
8/9	8	Solves and justifies a wide range of problems involving unfamiliar multiplicative situations including fractions and decimals, solves complex **proportional reasoning** problems, formally describe patterns in terms of general rules, solves complex, open-ended problems	*Proportional Reasoning* (by end of Year 8) – extending what is known beyond rule-based procedures to represent and solve problems involving fractions, decimals, percent, ratio, rate, and proportion
7/8	7	Compares, orders, sequences, represents, and renames whole numbers, fractions, decimals, and integers; appreciates inverse and identity relations, structure of place value system, recognises, describes and applies relationships between variables, algebraic processes, ratio; beginning to solve more complex proportion problems	
6/7	6	Extend decimal fractions, use partitioning strategies, more efficient, processes for dealing with all four operations, **proportion** problems, notion of variable, formally describe patterns	*(Equi) Partitioning* (by end of Year 6) – the missing link in building common fraction and decimal knowledge and confidence, constructs area and line models to compare, order, rename fractions and decimals
5/6	5	Uses **partitioning** strategies locate and rename fractions, extend place value larger whole numbers and tenths, flexible and efficient strategies of multiplying and dividing, area idea, Cartesian product problems, factors and multiples, strategies for adding and subtracting unlike fractions	
4/5	4	More efficient strategies for multiplying and dividing larger whole numbers, Tenths as a new place-value part, use partitioning strategies to compare fractions, 'for each' and times as many ideas to support **multiplicative thinking**	*Multiplicative Thinking* (by end of Year 4) – 'for each' and 'times as many' (factor) ideas key to developing efficient mental and written computation strategies and understanding rational number and in later years
3/4	3	Extended range of strategies, solves simple proportion problems, Cartesian product, thirding and fifthing partitioning strategies, key fraction generalisations, works with simple patterns	
2/3	2	More efficient strategies for counting large collections, array/region-based strategies for multiplication facts based on commutativity and distributivity, halving partitioning strategies to create fraction representations, key fraction generalisations, consolidating 2 and 3-digit **place value**	*Place Value* (by end of Year 2) – not all of place value, but a sense of structure, '10 of these is 1 of those, moves beyond counting by ones
1/2	1	**Trust the count** (part-part-whole knowledge), mental strategies for addition and subtraction, 2 and 3-digit place value, doubling/halving, array and region representations for multiplication, strategies for comprehending problems, explain and justifying solutions	*Trusting the Count* (by mid-Year 1) – flexible mental objects for the numbers 0 to 10, part-part-whole knowledge derived from subitising, composite unit