Teaching and Learning FOR, ABOUT and THROUGH problem solving

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"The usefulness of mathematics in solving problems in everyday life, work and further study has always been the justification for giving a central place in the curriculum. As a consequence of these considerations, school mathematics programs in Years 1 to 10 have tended to devote most time and energy to the acquisition of numerical concepts and computational skills. Measurement and spatial ideas have been given a lesser place, although they too are not inconsiderable in view of the important role they play in analysing and solving problems, particularly those of a practical nature.

Another basis for the centrality of mathematics in the school curriculum, and for the importance given to number and computation in particular, can be traced back to Plato's belief that arithmetic trained the mind to deal with abstractions. The persuasiveness of this belief has meant that the teaching and learning of mathematics has tended to evolve with very little regard for the real and tangible today, the unfortunate reality is that many students have not acquired the knowledge, skills and confidence that they need to solve practical, real-world problems, or to pursue further studies in mathematics.

... although most students are reasonably proficient with computational skills, the majority do not understand many basic mathematical concepts and are unable to apply the skills they have learned in simple problem situations" (Siemon, 1986, p. 10).

"The impression that comes through is that a lot (of students) get through byrote learning methods to solve a particular problem and as soon as you ask a question that requires some understanding of the mathematics, 50% of the candidates cannot do it" (Dr. Ken Sharpe, Chief Examiner HSC General Maths, 1995).

There have been a number of significant responses to this widely observed phenomenon: the Mathematics Curriculum and Teaching Program (Lovitt and Clarke, 1988), the new Mathematics Courses for the Victorian Certificate of Education, and "Exploring Mathematics in Classrooms" (EMIC), a professional development initiative of the Victorian Ministry of Education. Their message is clear: students must be given opportunities to construct their own knowledge based on understanding, and to develop the confidence and skills to apply that knowledge in unfamiliar situations. For example, these three kinds of learning activity are seen to underpin the work requirements for the new VCE Mathematics Study:

- "learning and practicing skills and applying them in standard situations;"
- "solving problems of an unfamiliar and non-standard kind, and using mathematical modelling as a tool in applying mathematical knowledge to real world problems;"
- "carrying out extended investigations projects using mathematical knowledge to investigate problems, situations or issues." (Fane, 1989)

The foundations for and requirements of such an expectation can be seen schematically in Figure 1.

In the past, most classrooms have tended to concentrate on only ONE aspect of this model, namely, learning and applying skills and applying them in standard situations, or teaching and learning FOR problem solving. The remaining outcomes were simply assumed to be the natural and inevitable consequence of such a focus. In other words, it was assumed that if students were trained rigorously in the basics, they would be able to solve problems in adult life and develop appropriate habits of mind.

Obviously, such a singular and exclusive focus is no longer justified. Having knowledge, skills, processes, and confidence to solve unfamiliar problems and carry out extended investigative projects cannot be acquired overnight. All three aspects, teaching and learning FOR, ABOUT and THROUGH problem solving, must be included simultaneously in the curriculum in a manner and proportion that is appropriate to the background, interests and needs of the students concerned. Proposed forms of alternative assessment make such an approach a manner of urgency.

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Figure 1

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Two problems

A bilder bag contains a collection of squares, pentagons and hexagons. Altogether there are 16 shapes in the bag. If there are 27 Activity 79 and 76 diagrams, how many of each shape are in the bag?

If you feel that you would like to attempt this task, but you are not sure how much of that you might do, then you have a problem. If you have no interest in the task, or if you already have access to a procedure or strategy which you are confident will produce the correct solution, then you do not have a problem at all. Notice that, having a problem is a very personal phenomena. It depends on who you are, what you know and how you feel about the task in question. For example, these three jelly snakes in each bag, how many jelly snakes are there altogether?

This is unlikely to be a problem for anyone reading this article, but it may well be a problem for a large number of children in their first year of school, who might well have an interest in solving this problem, but lack the means to do it directly.

A problem then, in a task or situation:

- that you want to or need to solve;
- that you believe you have some reasonable chance of solving, either individually or in a group, but
- for which, you, or the group, have no immediately available solution strategy.

Because a problem is such a personal experience it is impossible to say that one task is a problem while another is an exercise (a task for which a solution procedure is immediately available). For example, there is no evidence to suggest that 6354 is a problem for many grade four children, while how many handsakes are possible amongst a group of 5 people is not (Siemon, in preparation).

Nor does it make much sense to talk about good problems. Any problem may be rendered good, bad, or not a problem, depending on how

Teaching and learning FOR problem solving

Access to relevant knowledge, skills and strategies, can only be assured if these are established on the basis of understanding. The Knowledge and strategies used by children and adults in out-of-school settings (Carras, Carragher and Schleiman, 1985; Carragher, 1988) are often revealed more about what children have not learnt and understood at school than they do about the power and generalisability of such strategies to a range of other problem types. Clearly an understanding of numbers will always be fundamental to learning and using mathematics, and may itself provide a critique on, and understanding of, the computational procedures. Indeed, if mathematici
cal ideas are to be communicated effectively and if concepts and skills are to be taught without confusion, it is important that the notation ideas which give meaning to numbers be well established. Place value, regrouping and renaming are the fundamental concerns of numeration, and misconceptions with these have proven to represent the largest source of children's difficulties in arithmetic (Booker, 1987). Even when a child's errors appear in addition, subtraction, multiplication or division or with decimal fractions, the real source of the difficulty often lies in inadequate understanding of numeration.

Recently, a more general and more pervasive view of problem solving has become the focus of the mathematics syllabus. The use and interpretation of number and computational procedures that this demands means that the need for a properly established background of skill and understanding has become even more essential.

At the same time, the inclusion of particular problem-solving considerations has diminished the time available to develop purely arithmetic knowledge. The integration of calculator use from the earliest years might appear to offset this loss of time by requiring less highly developed skills but, concurrently, the need for well-understood processes is greatly increased. Not only the understanding which can direct the use of calculators and computers but also an ability to estimate results, to make approximate calculations and to check answers.

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Organising this development demands an analysis of the requirements for each concept and procedure. This has already been recognised, and in the past the calculus was largely given solely to the mathematical needs of the task. The basic components of each task were organised and presented in such a way that a new level was established only after the basic skills had already been met and learning proceeded from the simple to the more complex. However, while this type of analysis and organisation is necessary, it is not sufficient. Children do not simply receive knowledge as it is given and organised; they construct their own meanings for what is presented. If the order of presentation does not accord with their needs as children, they may easily form misconceptions related to their individual prior knowledge and understanding. Thus, it is necessary in planning the introduction of number ideas and computational abilities to consider the underlying mathematical understandings and how these are best developed by children. Materials are needed to bring to the base ten number system a meaningfulness that is linked to the names, direct the computational procedures and lead to a system of recording numbers and the algorithms that is understandable in terms of the mathematical ideas and the children's developing mathematical world.

Such an analysis shows that most of the understanding needed to process and structure the algorithms is drawn from number itself, that is, the skills and understanding needed to name, write, process and interpret numbers. It is important that children are able to consider numbers in a variety of ways: not just in terms of the counting with which number meaning begins, nor even with the system of place value which governs the construction and communication of numbers, but also in terms of the different ways in which a number can be renamed. For instance, 337 is five hundred 3 tens 7 ones; 8 hundreds 2 tens 17 ones 3 tens 2 ones; and 537 ones. Since a lot of learning of number ideas occurs informally, even before a child reaches school, it is necessary to consider the order and timing of the introduction of numerical concepts. Mathematics is the first subject area for children to be given a substantial grasp of counting and the numbers to ten. If a program limited only to numbers up to ten was to be introduced before children had been introduced to whole number concepts and ideas. In contrast, a lot of the first year of school has traditionally been given to activities which are concerned with logical activities such as order, sorting and classifying because a belief is necessary for the development of the number. Recent research (Clements and Callahan, 1983) has thrown doubt on this supposition, highlighting the importance of the development of an understanding of implicit cognitive tasks instead and revealing the strong link between logical activities and the development of general problem solving abilities.

While order of aspects of number ideas have been introduced at a later than optimal time, more frequently number ideas have been introduced too soon, too closely together or even not at all. It is for these reasons that children have experienced difficulties with number concepts, numbers with internal zeros, large numbers, the process of rounding and the use of inequality symbol. The size of the numbers and their order within the number sequence might seem an adequate basis for organizing their teaching, but it is even more important to consider the development of one new idea to the next, the regularity of the patterns in naming the numbers and the difficulties which could arise in recording and reading them. This has presented a different sequence to that traditionally used but one which experience has shown to be particularly smoothly effected with young children (Barry, Bookter, Perry and Siemon, 1965-80). This specific teaching of number ideas is a very necessary aspect of the development of computational skills, the consolidation of these skills also play a crucial part in bringing about a full realization of numeration.

Often, too little emphasis has been given to the concept for each operation, restricting the development to only one aspect or expression of the meaning, and focusing too early on the mastery of basic number combinations and written recording. While an ability to provide immediate, correct basic facts is crucial to completing computa- tions, an understanding of what each operation means, how it is to be interpreted and symbolised and which of action it represents is even more essential to using mathematics. Children need to be able to make the links between situations from their own experimental world to a language that describes the situations and symbols which express in mathematical terms. The language used needs to be extended over time to all of the expressions met in real situations that apply a particular operation as well as the formal expressions met solely in mathematical contexts.

When learning basic facts, attention needs to be paid to the strategies by which answers can be obtained. Some facts need to be allocated to knowledge which requires most attention. This means that pairs of facts such as 9 + 4 and 4 + 9 or 8 + 3 and 3 + 8 are learnt together while subtraction facts should build on known addition facts. Division facts should build on known multiplication facts rather than have two sets of related facts learnt in isolation. Only those facts that need to be learnt should be stressed; thus the tens facts should be recognised as knowledge acquired in numeration rather than treated as a separate entity to the earlier learning did not exist, and the tens facts in multiplication should be seen as a different expression of the doubles facts learnt in addition. Most importantly, not until children know how to obtain answers should a stress be placed on obtaining them quickly by rote. This is especially true because the answers that the teacher or learning situation (including games) requires then they must be reduced to guessing or learning by rote.

Only when children have acquired the concepts and basic facts for an operation it is possible to develop the algorithm for larger numbers. This development will be meaningful and logical if the materials are used to introduce the procedures, the language that is formulated to describe the use and the way in which these lead to the final recording of answers is taken from the initial concepts, through the basic facts to the symbolic procedures (see Fig. 2).

Subtraction, Multiplication and Division have a similar basis. An analysis of the understanding needed to develop computational abilities shows that there are fundamentally only two forms of thinking involved in all of the operations of arithmetic.

Finally, if should not be forgotten that a stress on calcula- tor usage and the approximations to answers that planning problem solving strategies often demands, will require a good ability to answer the problems. The mathematics that is required today to prepare children for problem solving requires that they are competent in pencil and paper calculation, and, more importantly, feel confident in choosing which is most appropriate to their particular needs in any given situation.
Applying the ASK-THINK-DO model in the mathematics classroom

Ultimately, one of the most important aspects of problem solving is the ability to develop and apply problem-solving strategies in the form which facilitates the use of the most powerful strategy available to the solver. Consider the following problem:

A farmer had sheep and cows in a paddock. If he counted 33 heads and 88 legs, how many sheep and how many cows were in the paddock? (HBJ 4/125)

A fourth-grade teacher can solve this problem using a variety of strategies (trial and error, draw a diagram, make a model, etc.) and thus get the answer by "their preferred method". It is the function of schooling to demonstrate the limitations and boundaries of such strategies and to challenge, modify, and extend those strategies. Just because children have efficient strategies for performing two digit multiplications in the market place does not mean that they have effective strategies and to challenge, modify and extend those strategies. Just because children have efficient strategies for dealing with division or the multiplication of decimals.

Translating an unfamiliar problem into a form that is amenable to the application of mathematical principles and procedures is a process described as problem solving. It is also important in relation to the acquisition of procedural knowledge, that is, knowledge about what to do, when, and how. The ASK-THINK-DO model of the problem-solving process developed for the HBJ Mathematics Series can be represented similarly.

and is intended to prompt the same sort of activity, that is, to encourage students to consciously link their conceptual and procedural knowledge by focusing on what is required, what is needed, how it can be obtained and how it relates to what is already known. In doing this, the model draws students attention to the problem-solving process and encourages them to accept responsibility for developing their own problem-solving behaviour. Recent research (Siemon, 1985, 1988) has established that overt attention to this in the classroom improves student problem-solving performance and confidence.

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the difference between 90 cents and the cost of the ice-cream. But this information is not given and two options need to be considered in order to completely answer the question. When confronted with this problem for the first time, one solver (E. Pick) has argued correctly for twenty minutes that "it wasn't mathematics because I didn't have one answer" (Siemon, in preparation). Presuming such problems, discussing them and analyzing them in terms of what can be asked, how many steps are involved, is there enough information, what needs to be done and so on, helps to develop an awareness of problem structure and facilitates the adoption of an appropriate plan. Some anecdotal evidence of this occurred when a grade 5 boy revisited his grade 4 classroom where he had participated in the discussion of the ice-cream problem some 16 months earlier. Seeing me in the classroom he spontaneously burst out with: "we did one just like the ice-cream one the other day... you had to do two answers!"

Encouraging children to THINK about what they know, what they can do and what they have found useful on previous occasions, by asking questions such as

What do I need to know?

How can I find out?

What have I done before?

What can I do that might help?

What is my plan?

Bm 17 in 2006

Does it look right?

Did the answer make sense?

Is there another way I could do it to make sure?

Have I thought of all possibilities?

Often this will necessitate a recycling through the ASK-THINK-DO phases of the problem-solving process. Other techniques that are used to help include: asking students to explain written reports describing what was done and why, posters, displays, oral presentations and problem posting. The latter is particularly effective as a means of assessing whether or not students have been working effectively. A technique, which is referred to as "Talking Heads" (Siemon, in preparation), has been found to be a very useful device for helping students reflect on their problem-solving behaviour. A blank circle and a speech or thought bubble is presented alongside a problem or task. Students are required to draw a face in the circle to indicate how they feel about the task and/or their performance and to record what they did, and how in the bubble. An example is presented below.

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leaves his birthday in line when it is the date of his birthday?

You try 4 on her 4th birthday.

The date was 2 days.

The sum of the 2 digits is 2.

You may be correct when you look by herself.

Mon Sun Tues Wed Thurs Fri Sat

Shahid bhai bhai bhai bhai

How long does the cold last for 326?

Shahid bhai bhai bhai bhai
Problem solving entails much more than "strategy games and puzzles", it is not "a topic" to be considered in isolation from the rest of the mathematics syllabus, or something which takes place only on Friday afternoons, during wet lunchtimes, in "task Centres" or during "Activity Maths". Children come to school each morning with pre-conceived ideas, or overwhelmed by unconsciousness from experience. It is acquired by conscious, critical reflection on past and immediate experience, the fact of which is self-evident; tasks, context and process. Teaching and learning ABOUT problem solving can and should be integrated into all aspects of the school mathematics program (Stenstrom, 1985). The ASK model of such a solving process can be applied to the following problem:

If the sheavers managed to shear 179 of the 368 sheep before lunch, how many sheep remained to be sheared that afternoon?

as effectively as it can be by one of the following:

Fred Frog dived into a well that was 12 metres deep. Unfortunately, the well was empty. Each day he could only jump 4 metres up the side of the well. Each night he slid down 2 metres. How many days does Fred take to get out of the well? (HBI 4/119)

Can you fill a marble along the blackboard ledge as quickly as you can hope to a distance of 10 metres? (HBI 4/68)

A 5 metre length of fencing costs $5 and each fence post costs $3. How much will it cost to make a triangular pen using 9 lengths of timber? (HBI 4/25)

How many ways can 4 postage stamps be attached so that they are joined on at least one edge? (HBI 6/257)

Should boys between the ages of 5 and 16 be banned from riding bikes? (HBI 5/292)

Problems such as these provide opportunities to apply appropriate knowledge, skills and strategies, but even more importantly, they provide opportunities to examine problem structure and reflect on the problem solving process. Teaching and learning ABOUT problem solving encourages students to recognize and accept responsibility for their problem solving by providing them with the knowledge and techniques to access, monitor and direct what they know and what they do.

Teaching and learning THROUGH problem solving

Problem solving may be seen as an end in itself, the natural focus of mathematics, which is based on real-life needs and applications. However, this is only one of the justifications for making problem solving the center of activity in school mathematics programs. Problem solving recontextualizes mathematics. It provides a rich background against which to interpret meaning, to apply basic ideas, to explore strategies, and to evaluate personal performance. It is much more likely to lead to insights into the processes of mathematical thinking than traditional approaches which treat mathematics as a set of isolated procedures to be learned more or less by rote. In other words, new mathematical ideas can grow out of appropriately posed mathematical problems. It is in this sense that mathematics can be said to be useful and learnt THROUGH problem solving.

In early years of primary school, this has always happened. Stories and problems from the children's environment are posed to help develop the concepts associated with number and the four operations. Children's toys, familiar objects and events and other more structured materials are then used to develop a more generalized understanding of the concept and, eventually, the associated recorded forms. A concept is said to be grasped when the match between story problems, materials and symbols is internalized.

ACTION STORY

MATERIALS

SYMBOLS

While the initial learning is arrived at THROUGH problem solving, it is consolidated, extended and applied by teaching and learning FOR and ABOUT problem solving. For example, although the fundamental concepts associated with addition, subtraction, multiplication and division are established and linked to words and symbols through problem solving, their successful application to more generalized problems depends upon the construction of standard meanings and processes. Once these have been established and consolidated through practice and application, usually by the later primary years, the potential for further mathematical insights through problem solving is restored and new ideas, which build on these foundations, can emerge. Unfortunately, this potential is not always realized, as evidenced by traditional approaches to the teaching and learning of algebra which do not recognize or begin with the understandings the child has arrived at after 6 or 7 years of schooling. Little wonder that children fail to grasp the significance of the new expressions in terms of the more abstract and general reasoning processes that they allow.

Algebra involves the extension of general solution procedures to identified classes of problems which have essentially the same result. A sequence of experiences which leads from concrete arithmetic situations to algebraic generalizations must establish that the use of letters is a powerful means to express such results. A first use is simply as labels to identify the objects being examined and grow naturally out of words used to describe them in a manner analogous to the use of letters in measurement, for example, l for length or W for width. When this has been established and accepted, relationships between the objects which have been identified and labelled can also be expressed using the letters that have provided the labels, as algebraic generalizations for the relationships given by $l = A \times L \times W$. The use of tables of values to show these relationships can then in turn suggest more concise ways of expressing the results by means of the number which identifies a particular entry (see Fig. 2).

<table>
<thead>
<tr>
<th>Start</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>12</td>
<td>50</td>
</tr>
<tr>
<td>25</td>
<td>75</td>
</tr>
<tr>
<td>32</td>
<td>160</td>
</tr>
</tbody>
</table>

In this way, the use of letters to express relationships occurs somewhat naturally and provides the basis for using the letters themselves to find and verify patterns (see Fig. 4).

![Diagram of Choose a number, Double it, Add 3, subtract the original number, Add your age, multiply by 2, Add your original number, Double it again, Add 3, subtract the original number, Subtract by 3, divide by 2, Add your age number](HBI 7/222)

Only when the development of a generalised arithmetical has established the need for and power of algebraic symbols can algebra be extended to a topic in its own right and meaningful procedures for manipulating the symbols be considered. So while the conceptual basis is arrived at through problem solving, its extension and consolidation is achieved through a mix of teaching and learning FOR and ABOUT THROUGH problem solving.

Teaching and learning THROUGH problem solving can be a springboard to further learning in geometry and measurement also. For example:

The electricity supply to a new housing estate is to be put underground. The SEC wants to connect the houses using the shortest path to minimise cost and power loss. What is the shortest route amongst all houses?

A shoe manufacturer can produce 10,000 pairs of fashion shoes per month. If the company produces shoes in sizes 1 to 12 and in fittings A, B, C, D and E, how many of each size and fitting should be produced to minimise waste and maximise profit?

Summary

In conclusion, while it has been argued elsewhere that "Problem solving can be both medium and message; it can be both content and context; and we believe it is at its most effective when provided the structure for learning" (Lovitt and Clarke, 1988, p.489), this implies that only two aspects of the model described earlier are required to promote problem solving behaviour, namely, teaching and learning ABOUT problem solving and teaching THROUGH problem solving (see Fig. 1). But teaching and learning FOR problem solving is just as important and just as necessary to successful problem solving. Without the appropriate knowledge, skills and processes, meaning cannot be constructed and problem solving cannot proceed. While it is undoubtedly true that the "most elaborate and extensive toolkit is of little use if it remains untried" (Lovitt and Clarke, 1988, p.407), it is also true that the most elaborate and extensive set of keys is of little use if the tool box is empty.
Teaching and learning FOR problem solving is needed to ensure the availability of appropriate knowledge, skills and strategies, built-in understanding and exercised with confidence.

Teaching and learning ABOUT problem solving is needed to provide the means to access, monitor, direct what is known and what can be done.

Teaching and learning THROUGH problem solving is needed to provide a context for further learning and to exercise the application of the knowledge, skills and processes acquired as a result of the first two approaches. Each approach has a vital and critical role to play in the acquisition of sophistication in the thinking of all levels. We simply cannot afford to concentrate on any one or two at the expense of all three.

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"Look — if you have five pocket calculators and I take two away, how many have you got left?"