THE DERIVATION OF A LEARNING ASSESSMENT FRAMEWORK FOR MULTIPLICATIVE THINKING

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Research has shown that many students in Years 5 to 9 experience considerable difficulty with rational number, algebra, and the application of multiplication and division to a broader range of problem types. While each of these has been the focus of considerable research, there is little advice about how these ideas are connected and jointly develop over time. This paper describes the genesis of a learning assessment framework for multiplicative thinking based on Rasch analysis of student responses to a range of rich tasks to inform more targeted approaches to teaching mathematics in the middle years.

INTRODUCTION

The purpose of this paper is to share some of the findings from a current research project aimed at scaffolding numeracy learning in the middle years of schooling. The project was prompted by the results of an earlier study which indicated that many students in Years 5 to 9 have difficulty with what might broadly be described as multiplicative thinking. That is, thinking that is characterised by (i) a capacity to work flexibly and efficiently with an extended range of numbers (for example, larger whole numbers, decimals, common fractions, and/or per cent), (ii) an ability to recognise and solve a range of problems involving multiplication or division including direct and indirect proportion, and (iii) the means to communicate this effectively in a variety of ways (for example, words, diagrams, symbolic expressions, and written algorithms). In particular, the results suggest that while most students are able to solve multiplication problems involving relatively small whole numbers they rely on additive strategies to solve more complex multiplicative problems involving larger whole numbers, rational numbers, and/or situations not easily modelled in terms of equal groups (Siemon & Virgona, 2001). This suggests that the transition from additive to multiplicative thinking is nowhere near as smooth or as straightforward as most curriculum documents seem to imply, and that access to multiplicative thinking as it is described here represents a real and persistent barrier to many students' mathematical progress in the middle years of schooling.

This observation is supported by research more generally. For example, there is a considerable body of research pointing to the difficulties students experience with multiplication and division (Mulligan & Mitchelmore, 1997; Anghileri, 1999), and the relatively long period of time needed to develop these ideas (Clark & Kamii, 1996; Sullivan, Clarke, Cheeseman & Mulligan, 2001). Student's difficulties with rational number and proportional reasoning have also been well documented (for example, Hart, 1981; Harel & Confrey, 1994; Lamon, 1996; Baturo, 1997; Misailidou & Williams, 2003). Moreover, there is a growing body of research

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documenting the link between multiplicative thinking and rational number ideas (Harel & Confrey, 1994; Baturo, 1997); multiplicative thinking and spatial ideas (Battista, 1999), and the importance of both as a basis for understanding algebra (Gray & Tall, 1994). While this work contributes to a better understanding of the 'big ideas' involved, very little is specifically concerned with how these ideas relate to one another and which aspects might be needed when, to support new learning both within and between these different domains of multiplicative thinking. Moreover, very little of this work appears to be represented in a form and language that directly translates to practice in the middle years of schooling.

Ball (2000) identified three problems that need to be solved if we are "to prepare teachers who not only know content but can make use of it to help all students learn" (p.6). First, she suggests, we need to re-examine what content knowledge matters for good teaching. Second, we need to understand how subject matter must be understood to be useable in teaching (pedagogical content knowledge), and thirdly we need "to create opportunities for learning subject matter that would enable teachers not only to know, but to learn to use what they know in the varied contexts of practice" (p.8). Simon's (1995) idea of constructing hypothetical learning trajectories as mini-theories of student learning in particular domains appears to offer a useful approach to these problems as they provide an accessible framework for identifying where students 'are at' and offer starting points for teaching.

In Australia, learning trajectories have tended to take the form of learning and assessment frameworks which have been developed and validated in terms of a number of discrete domains such as counting, place-value, and addition in the early years of schooling (Clarke, Sullivan, Cheeseman & Clarke, 2000). While learning and assessment frameworks for multiplication and division have been developed at this level, the evidence suggests that very few students in Years P to 3 are at the point of abstracting multiplicative thinking, that is, able to work confidently and efficiently with multiplicative thinking in the absence of physical models (Mulligan & Mitchelmore, 1997; Sullivan et al., 2001). Where teachers were supported to identify and interpret student learning needs in terms of the frameworks, it was shown that they were more informed about where to start teaching, and better able to scaffold their students' mathematical learning (Clarke, 2001). This suggests that clarifying and extending the key ideas involved in multiplicative thinking and working with teachers to identify student learning needs and plan targeted teaching interventions, is likely to contribute to enhanced learning outcomes for students in the middle years.

THE PROJECT

The research reported in this paper is a key part of a larger study aimed at investigating the efficacy of a new assessment-guided approach to improving student numeracy outcomes in Years 4 to 8. In particular, it is concerned with documenting the development of multiplicative thinking which is known to be a major barrier to students' mathematical progress in the middle years. The research is premised on the

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view that where teachers have a clear understanding of learning trajectories and where students 'are at' in terms of those trajectories, they are better able to make informed decisions about what targets to set and how to achieve them. A major component of the research study was the identification of an evidence-based learning and assessment framework which could be used to support a more targeted approach to the development of multiplicative thinking in the middle years of schooling.

While the project was designed to address a number of research questions, the one that will be addressed here is the extent to which it is possible to identify and validate an integrated learning assessment framework for multiplicative thinking that relates to and builds on what is known in the early years.

METHOD:

The research plan was designed in terms of three overlapping phases. Phase 1 was aimed at identifying a broad hypothetical learning trajectory (HLT) which would form the basis of the proposed learning and assessment framework for multiplicative thinking (LAF). Phase 2 involved the design, trial and subsequent use of a range of rich assessment tasks which were variously used at the beginning and end of the project to inform the development of the LAF. Phase 3 involved research school teachers and members of the research team in an eighteen month action research study that progressively explored a range of targeted teaching interventions aimed at scaffolding student learning in terms of the LAF.

Just over 1500 Year 4 to 8 students and their teachers from three *research* school clusters, each comprising three to six primary (K-6) schools and one secondary (7-12) school, were involved in Phases 2 and 3 of the project. A similar group of Year 4 to 8 students from three *reference* school clusters was involved in Phase 2 only.

Phase 1:

The initial HLT was derived from a synthesis of the research literature on students' understanding of multiplicative thinking, proportional reasoning, decimal place-value and rational number. It comprised nine 'levels' of increasingly complex ideas and strategies grouped together more on the basis of 'what seemed to go with what' than any real empirical evidence, although this was used where available. The HLT was then used to select, modify and/or design a range of rich tasks including at least two extended tasks (Callingham & Griffin, 2000; Siemon & Stephens, 2001). The tasks were trialled and either accepted, rejected or further modified on the basis of their accessibility to the cohort, discriminability and perceived validity in terms of the constructs being assessed. Trial data were used to develop scoring rubrics and feedback from the trial teachers was used to modify the assessment protocol.

Phase 2

A total of just under 3200 students in the research and reference schools completed the initial assessment tasks in May 2004. To control order effects and maximise the number of tasks that could be included, four different test booklets were prepared. Each test comprised one of two extended tasks (9 or 13 items) and five shorter tasks (2 to 4 items each). Common tasks were variously used to link the four tests. For instance, two short tasks were completed by all students, two more were completed by 75% of the students and another two were completed by 50% of the students.

Research school teachers administered the tasks and scored these on the basis of the scoring rubrics provided. A professional development session was provided to support this process at a meeting of all research school teachers at the beginning of 2004. Reference school teachers were briefed on the purpose and administration of the tasks at a separate meeting but were not required to score students' work. This was undertaken by a group of scorers under the direction of the research team. Following this, and to support the further elaboration of some levels of the LAF particularly those hypothesised at the upper end of the framework, a number of additional tasks were developed and trialled in October 2004. The results of this exercise and the subsequent assessment of Year 7 students in March 2005 were used to inform the development of the LAF.

Analysis

The data were analysed using Rasch (1980) measurement techniques, which allowed both students' performances and item difficulties to be measured using the same log-odds unit (the logit), and placed on an interval scale. Consistent with other studies that have used Rasch to evaluate mathematical performance, for example, Izard, Haines, Crouch, Houston and Neill (2003), Misailidou & Williams (2003) and Watson, Kelly, & Izard (2004), anchoring strategies were used to place students on a common achievement scale. The Quest computer program (Adams & Khoo, 1996) was used to apply the Partial Credit Model (Masters, 1982) and obtain a variable map showing the placement of students and items along the scale. The Quest program evaluates the fit of the data to the Rasch Model: the default acceptable values (between 0.77 and 1.3) were used to check the fit of the data to the model. The values of the Separation Reliability for both items and persons were high, indicating consistent behaviours of both items and persons.

An excerpt from the variable map produced for one version of the initial assessment tasks is shown in Figure 1. Students are shown (anonymously) on the left-hand side of the variable map (in this case, each x represents 2 students). The coded items on the right refer to a particular part of each task, for example, pkpb.2 (highlighted in bold) refers to part b of the *Packing Pots* task. The location of the coded item indicates the point at which students scoring at this level have a 50% chance of satisfying the scoring criterion indicated by the number following the full stop (in this case, a score of 2 out of a possible 0, 1 or 2). The logit value at this point is referred to as the item threshold (in this case, -0.16).

The variable maps for each test administration were combined to produce an overall list of item thresholds which differentiated items on the basis of student performance.

Easier, more accessible items had relatively low item thresholds. For example, the item threshold associated with a score of 1 for part b of the *Tables and Chairs* task (possible scores 0 or 1) was –2.69. The item threshold associated with a score of 4 on part b of the *Adventure Camp* task (possible scores 0, 1, 2, 3 or 4) was 3.53.

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            XXXXXXXXXXXXXXX  |           bthf.2    bthh.3 bthi.3 pzpc.2 msna.2 adcb.2
 1.0            XXXXXXXXXXX  |                                   cncb.2
              XXXXXXXXXXXXX  |                      bthi.1 bthi.2        msnb.2
            XXXXXXXXXXXXXXX  |         bthe.2       pkpa.2 cnca.2
                   XXXXXXXX  |  bthd.2                       pkpc.2 pzpa.2 pzpb.2 msna.1
               XXXXXXXXXXXX  | bthc.2       bthg.1 bthh.2
 0.0            XXXXXXXXXXX  |                    bthh.1               pkpd.1
                 XXXXXXXXXX  |           bthf.1          pkpb.2
                     XXXXXX  |                                       cncb.1
                     XXXXXX  |                                    pzpc.1              msnb.1
                    XXXXXXX  |
                      XXXXX  |  bthd.1 bthe.1       pkpc.1 pzpa.1
-1.0              XXXXXXXXX  |                                       pzpb.1
                        XXX  |                                      cnca.1
                        XXX  |                                      adca.1 adcb.1
                       XXXX  | bthc.1
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Figure 1. Excerpt from Variable Map V1 (N=358, full logit range: -3 to 4.2)

THE FRAMEWORK

A detailed content analysis of items led to the identification of eight relatively discrete categories which described what students might be expected to be able to do if they scored within the corresponding band of item thresholds. For instance, in terms of the items shown in Figure 1, items with thresholds ranging from 0 to 0.5 were grouped together (with other items not shown on this version) to form a discrete category that might broadly be described as early multiplicative thinking (see Level 4 in Table 1 below). Students who scored within this band were generally able to solve simple 2-digit by 2-digit multiplication problems (for example, a snail travels at 15 cm/minute, how far will it travel in 34 minutes), and accurately represent and describe the results of sharing 4 pizzas among 3 and 3 pizzas among 4, but they had some difficulty interpreting remainders and justifying their responses.

Level 1: Solves simple multiplication and division problems involving relatively small whole numbers but tends to rely on drawing, models and count-all strategies. May use skip counting for groups less than 5. Makes simple observations from data and extends simple number patterns. Multiplicative thinking (MT) not really apparent as no indication that groups are perceived as composite units, dealt with systematically, or that the number of groups can be manipulated to support more efficient calculation
Level 2: Counts large collections efficiently, keeps track of count but needs to see all groups. Shares collections equally. Recognises small numbers as composite units (eg, can count equal groups, skip count by twos, threes and fives). Recognises multiplication needed but tends not to be able to follow this through to solution. Lists some of the options in simple Cartesian product situations. Some evidence of MT as equal groups/shares seen as entities that can be counted.

Level 3: Demonstrates intuitive sense of proportion. Works with useful numbers such as 2 and 5 and intuitive strategies to count/compare groups (eg, doubling, or repeated halving to compare simple fractions). May list all options in a simple Cartesian product, but cannot explain or justify solutions. Beginning to work with larger whole numbers and patterns but tends to rely on count all methods or additive thinking (AT).
Level 4: Solves simple multiplication and division problems involving two-digit numbers. Tends to rely on AT, drawings and/or informal strategies to tackle problems involving larger numbers, decimals and/or less familiar situations. Tends not to explain thinking or indicate working. Partitions given number or quantity into equal parts and describes part formally. Beginning to work with simple proportion.
Level 5: Solves whole number proportion and array problems systematically. Solves simple, 2-step problems using a recognised rule/relationship but finds this difficult for larger numbers. Determines all options in Cartesian product situations involving relatively small numbers, but tends to do this additively. Beginning to work with decimal numbers and percent. Some evidence MT being used to support partitioning. Beginning to approach a broader range of multiplicative situations more systematically
Level 6: Systematically lists/determines the number of options in Cartesian product situation. Solves a broader range of multiplication and division problems involving 2-digit numbers, patterns and/or proportion but may not be able to explain or justify solution strategy. Renames and compares fractions in the halving family, uses partitioning strategies to locate simple fractions. Developing sense of proportion, but unable to explain or justify thinking. Developing capacity to work mentally with multiplication and division facts
Level 7: Solves and explains one-step problems involving multiplication and division with whole numbers using informal strategies and/or formal recording. Solves and explains solutions to problems involving simple patterns, percent and proportion. May not be able to show working and/or explain strategies for situations involving larger numbers or less familiar problems. Constructs/locates fractions using efficient partitioning strategies. Beginning to make connections between problems and solution strategies and how to communicate this mathematically
Level 8: Uses appropriate representations, language and symbols to solve and justify a wide range of problems involving unfamiliar multiplicative situations, fractions and decimals. Can justify partitioning, and formally describe patterns in terms of general rules. Beginning to work more systematically with complex, open-ended problems.

Table 1. Summary Learning Assessment Framework for Multiplicative Thinking

CONCLUDING REMARKS

The detailed item response analysis generally supported the developmental sequence reflected in the initial HLT, but the richer descriptions generated by the analysis prompted a reduction in the number of categories and revealed some interesting anomalies. For instance, while the item response data confirmed that partitive division is generally more accessible than quotitive division (Greer, 1992), and discrete quantities such as *24 pots*, were generally easier to work with than continuous ones such as *34 metres* (Hart, 1981), the relatively late emergence of efficient partitioning strategies to represent or locate common fractions and decimals (Levels 6 and 7 of the Framework) suggests that the link between division and fractions and the issue of discrete and continuous might be much more complex than previously imagined.

The primary purpose of this paper was to describe the derivation of a Learning Assessment Framework for Multiplicative Thinking that is being used to help scaffold student learning in this area. While it is too early to speculate on the outcomes that may or may not emerge as a consequence of the action research component of the current study (Phase 3), classroom observations and teacher feedback to date are promising, suggesting that the LAF is a powerful tool in helping teachers identify specific learning needs and plan appropriate teaching responses. Subsequent analyses will address whether the teaching responses were effective.

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