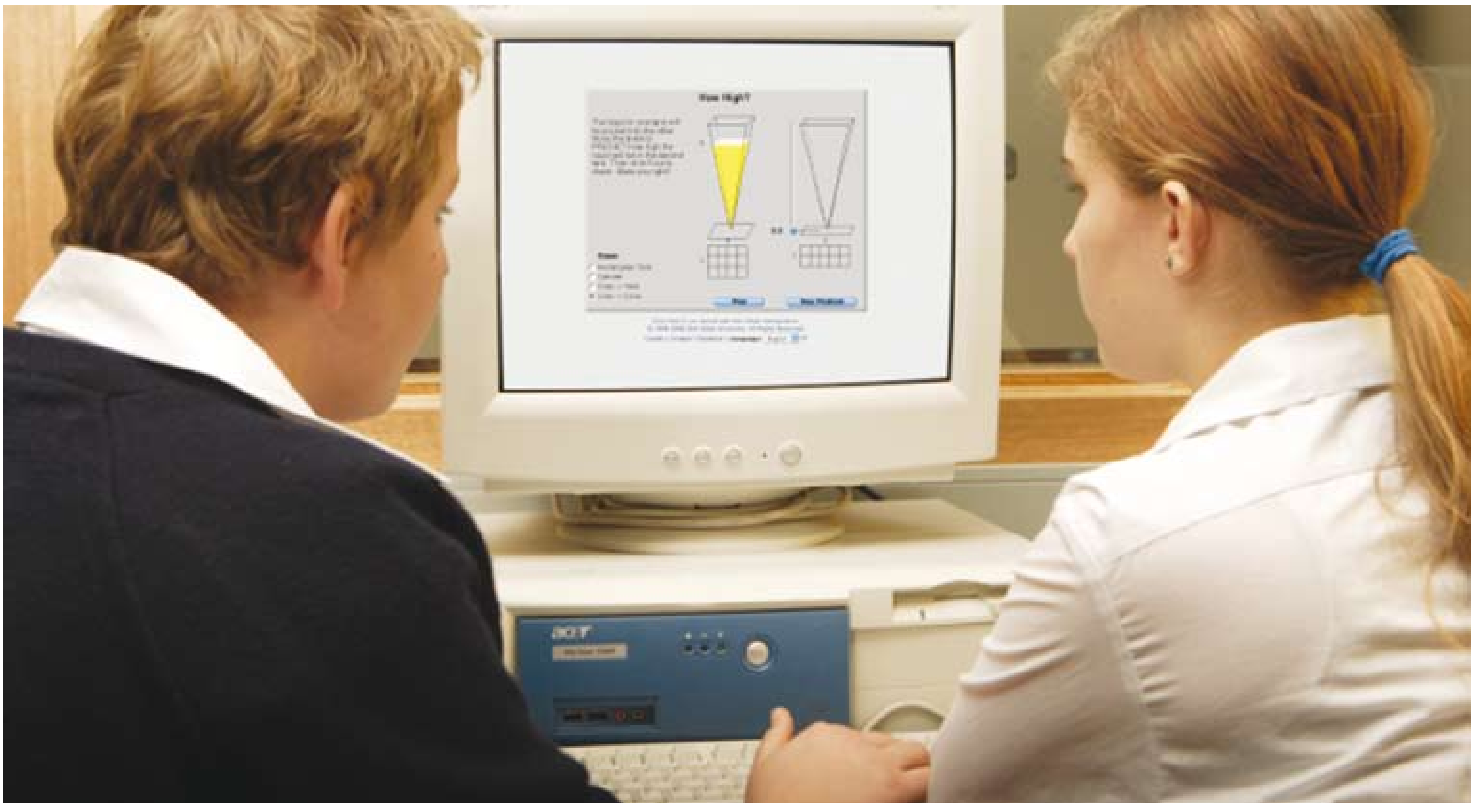

Teaching Secondary Mathematics

Module 5

Understanding students' mathematical thinking: Focus on algebra and the meaning of letters

5

State Government Victoria
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Contents

Introduction to Module 5: Understanding students' mathematical thinking: Focus on algebra and the meaning of letters

The Teaching Secondary Mathematics Resource provides support and guidelines for effective practice for classroom teachers and school leaders of mathematics, especially from Years 7–10.

This resource is for:

- all secondary mathematics classroom teachers to deepen their understanding of mathematics. This will inform their planning for mathematics and highlight opportunities for assessment of mathematics in other domains of the Victorian Essential Learning Standards (VELS)
- mathematics leaders in schools to plan opportunities for professional learning for the teachers of mathematics, in professional learning teams and/or for individual teachers
- differentiating the professional learning needs of mathematics teachers in schools.

Use of this module

This module allows for flexibility in modes of engagement with professional learning. The module booklet needs to be used in conjunction with the PowerPoint slides accompanying this resource.

Workshop approach

The materials of this module can be used by a presenter in a workshop for a school or a cluster of schools. A presenter, appointed from within or outside a school or cluster, is responsible for preparing presentations, facilitating discussions and outlining processes for collaborative planning.

Where a group is working collaboratively through these modules, a designated area is required for participants to share ideas, stories and samples in a climate of mutual respect. Regular after school meetings in a particular venue, such as the library, create a productive sense of community.

Individual use

The materials of this module are also suitable for private study and reflection. Individual users become both 'presenter' and 'participant'. While they are not able to engage in group discussions or whole-school planning, individual users can readily adapt the suggested group discussions and whole-school planning activities to private reflection, writing and classroom planning.

It is suggested that individuals identify a colleague or a buddy with whom to share their thoughts and to support the development of their understandings through ongoing dialogue. Individuals may complete all the modules or choose a combination depending on their interests or needs.

Web connections

The 'Teaching for Secondary Mathematics' resource is located at http://www.education.vic.gov.au/studentlearning/teachingresources/maths/teachsec/default.htm.

Before commencing to plan any elements of the program, schools are strongly advised to visit the Mathematics Domain page to review the most up-to-date advice, resources and information relevant to each module of the program. Many elements of this resource are available online in a downloadable format. There are links to assist schools to locate relevant information.

Mathematics Domain (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/default.htm)

See the website for further details about this additional information or contact the student learning help desk on studentlearning@edumail.vic.gov.au

Content of the module

The module comprises Module 5 booklet and the accompanying slide presentations which can be downloaded from http://www.education.vic.gov.au/studentlearning/teachingresources/maths/teachsec/module5.htm

The following are included in this document:

- the **User's Guide** that assists the user through the professional learning program
- hard copies of the **slide presentations** and **resource sheets**
- selected **resources.**

Organisation of the module

Computer access is required for all modules. If a group is completing the modules, a data projector and tables that enable people to sit together and work collaboratively are also necessary. The presenter should encourage participants to raise questions throughout the ensuing presentation. This presentation should take approximately one hour, depending on the depth of discussion and types of activities that facilitators incorporate..

Required resources

- Indicator of progress: The meaning of letters in algebra: 4.25 (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/structure/St42504P.htm)
- Mathematics Development Continuum – Mapping the Indicators of Progress (http://www.eduweb.vic.gov.au/edulibrary/public/teachlearn/student/mathscontinuum/indicatorsgrid.pdf)
- Developmental Overview of Structure (http://www.eduweb.vic.gov.au/edulibrary/public/teachlearn/student/mathscontinuum/mcdostructure.pdf)

Icons

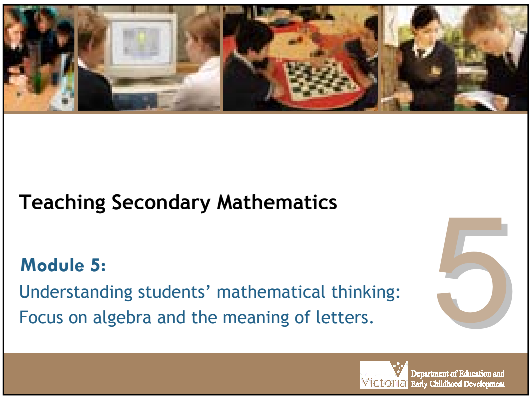
The following icons have been used in this workshop program:

Distribute handout: 

Group discussion: 

Group activity: 

User's Guide to Module 5: Understanding students' mathematical thinking: Focus on algebra and the meaning of letters



Slide 1: Title slide

Introduction

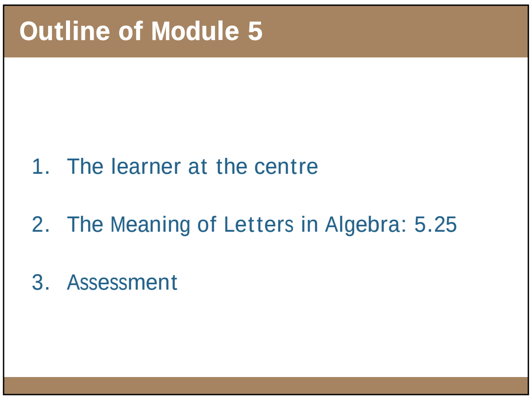
Slide 1 is the title slide

There are two themes running concurrently in this module:

- the pedagogical theme of understanding students' thinking in algebra
- the mathematical theme of the meaning of letters in algebra.

This module has been developed from the examples provided in the indicator of progress: 'The meaning of letters in algebra: 4.25' which is found on the Mathematics Developmental Continuum P–10.

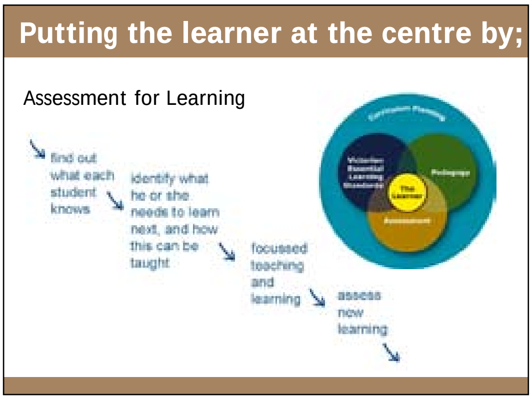
See Indicator of progress: The meaning of letters in algebra: 4.25 (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/structure/St42504P.htm)



Slide 2: Outline of Module 5

Outline of the module

Slide 2 provides an outline of this module. Participants will be completing activities to help them understand students' mathematical thinking in algebra.



Slide 3: Putting the learner at the centre by

Putting the learner at the centre by 'assessment for learning'

Slide 3 explains the need for assessment for learning. Research shows that assessment for learning makes a difference to student learning. It is a cycle which aims to match instruction to a learner's needs. The cycle begins with finding out what each student knows.

The indicators of progress are points on the learning continuum that highlight critical understandings required by students in order to progress through the standards. This information is used in a formative way, providing a means to identify what the student needs to learn next. A teacher could then select activities that focus on what the student needs. After teaching and learning, assessment can be used to decide if a student revisits this material or moves on.

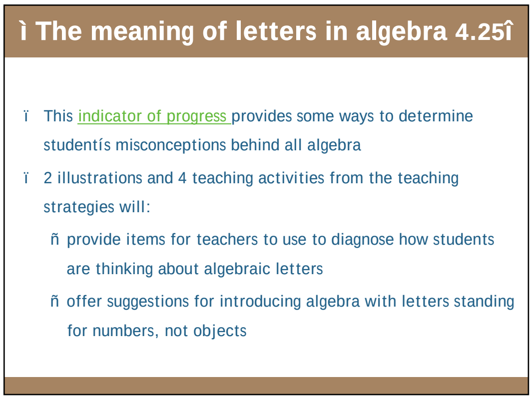
Student thinking about the meaning of letters in algebra

Slide 4 explains the different components of this indicator of progress. It describes ways to determine students' misconceptions behind all algebra. It describes activities and classroom ideas.

Provide participants with a copy of this indicator of progress:

See Indicator of progress: The meaning of letters in algebra: 4.25 (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/structure/St42504P.htm)

Provide participants with an opportunity to investigate this document. Draw attention to the illustrations and activities which provide ways to determine students' misconceptions regarding algebra.



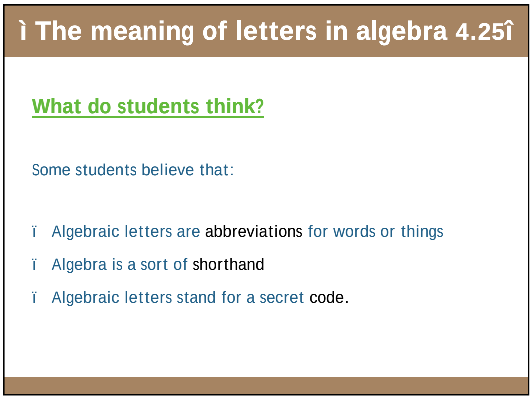
Slide 4: The meaning of letters in algebra: 4.25 — indicator of progress

What do students think?

Slide 5 describes student thinking about algebra.

Research around the world has shown that students draw on what they know of other symbol systems when they are trying to understand symbols in algebra. Common errors in interpreting letters can be explained by:

- Students drawing analogies with other symbol systems (e.g. letters in algebra are abbreviations or initials for things). In other symbol systems letters are often used as abbreviations for words in everyday life.
- Misleading teaching materials (e.g. fruit salad algebra and variations) which give them the idea that algebra is a kind of shorthand.
- Students trying to write mathematical ideas that cannot easily be written in algebra (e.g. trying to say in symbols that when x increases by 1, y increases by 4).
- Students thinking that algebraic letters stand for a secret code. 'Letter as a code' is an easier misconception to remove compared with letter as an object, which is bound to a deeper level of conceptual understanding.



Slide 5: The meaning of letters in algebra: 4.25 — What do students think?

Illustration: Algebraic letters do not stand for things

Slide 6 illustrates how students can easily develop the misconception that letters stand for objects and not numbers.

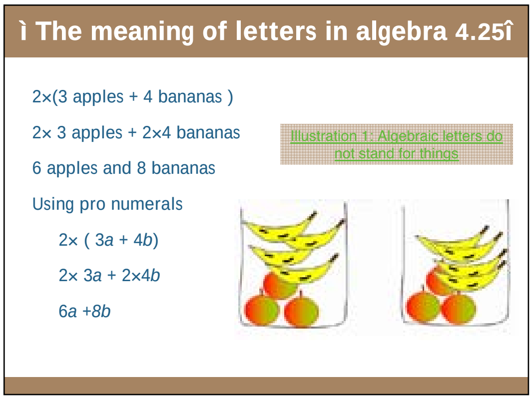
Read, or provide the opportunity for the participants to read through the explanation on slide 6, but emphasise the word **pronumeral**, where in this instance it represents fruit, not numbers.

Use slides 6–8: Fruit salad algebra

Invite participants to discuss the following question:

- 'If we have two bags, each containing 3 apples and four bananas, how many apples and bananas do we have?'

Slide 7 and 8 follows the scenario given on slide 6. These slides describe the reasons for developing this misconception.



Slide 6: The meaning of letters in algebra: 4.25 — Illustration

ì The meaning of letters in algebra 4.25î

Students may have developed these misconceptions from:

Copying other symbol systems letters are often used as:

ï Abbreviations for words in everyday life

ï Teaching ñfruit salad algebra

ï False analogies e.g. with codes

Slide 7: The meaning of letters in algebra: 4.25 — misconception

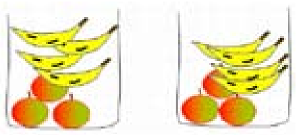
ì The meaning of letters in algebra 4.25î

Anything wrong with this reasoning?

2 x (3 apples + 4 bananas)
= 2 x 3 apples + 2 x 4 bananas
= 6 apples + 8 bananas

Using pronumerals

$2 \times (3a + 4b)$
$= 2 \times 3a + 2 \times 4b$
$= 6a + 8b$



Slide 8: The meaning of letters in algebra: 4.25 — misconception

ì The meaning of letters in algebra 4.25î

Confusions about the meanings of letters: doughnuts

Students were asked the following question:

Write an equation which describes the situation
ª ì 6 doughnuts cost 12 dollarsî

Correct equation, written by nearly all students,
ª 6d = 12, but what does it mean?

Instructions to students:
After you have written the equation, say what quantity each of the numerals and pronumerals represents.

Slide 9: The meaning of letters in algebra: 4.25 — confusion about the meaning of letters

The key message is that **letters should stand for numbers**.

Fruit salad algebra of the type shown has occurred frequently in textbooks, reinforcing students' misconception that a letter stands for an object. In this example, 'a' and 'b' are clearly being used as abbreviations for an apple and a banana rather than a pronumeral to represent the number of apples and the number of bananas. It is this type of example that contributes to students' misunderstanding in algebra.

In fact, the algebraic letters illustrated on this slide do not stand for numbers. In these equations the expression $4b$ does not mean 4 multiplied by the number b (as it does in algebra). It means 4 bananas!!!!. This is **shorthand** – a summary description of the picture that looks like algebra, but it is not algebra. The slide illustrates that teaching may unintentionally promote a misconception in mathematics.

Donuts: student misconceptions about letters as objects

Slide 9 provides another example to illustrate student confusion about the meaning of letters. It illustrates some of the errors that students make when they are unclear about the meanings of algebraic letters and equations. Participants can access this example through:

- Indicator of progress: The meaning of letters in algebra: 4.25 – Activity 3: Confusions about the meaning of letters (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/structure/St42504P.htm#a2)

In this instance, this example describes a situation where a teacher has asked the students to write an equation which describes the situation '6 doughnuts cost 12 dollars'. In this class, the correct equation, written by nearly all students, is $6d = 12$. However ask the participants what this means.

To clarify their understanding, students were given a further instruction 'After you have written the equation, say what quantity each of the numerals and pronumerals represents'.

Use slides 9–12: Donut problem

Invite the participants to answer the following questions relating to the following problem:

- What do the letters and numbers represent?

Participants should answer:

6 – the number of doughnuts

d – cost per doughnut,

12 – the total cost

Ask the participants to:

- Write some of the things that students might think these pronumerals represent.
- Write students' incorrect equations, if any.

Student responses to the question

Slide 10 provides the student responses to the questions.

Compare student responses with participants' answers. Participants should note that most of the students' responses are incorrect!

Students think 'd' stands for doughnuts – the object or word, not a number!

Only Cath has provided the correct response (d is the cost per doughnut).

Analyses of the student responses

Use Slides 11 and 12 to stimulate group discussion through the analysis of incorrect student responses.

Ask the participants to explain what is wrong with the student responses and why this might be significant.

Participant responses may include:

> Anna, Dan, and Ellie perceive d as objects (ie doughnuts).
>
> Ben seems to have misunderstood the question: he thinks he just needs to label the numbers as either numerals or pronumerals.
>
> Dan writes the problem in shorthand in the form of the equation.
>
> Ellie is nearly there but has got confused with d being a doughnut.

An unusual response

Slide 12 examines 'Fran's' response. Explain to participants that this is an unusual response and a 'wrong' equation. However, Fran has thought carefully about the meaning of what she has written. She has actually mentally interpreted and solved the given problem.

Wanting d to be the number of doughnuts (ie 6), Fran wrote the equation $2d = 12$ and interpreted it correctly.

Fran did not answer the question correctly, but her reasoning was much better than nearly all the other students who wrote $6d = 12$ without understanding what the equation actually meant.

Interpreting student work 6d=12

ì 6 doughnuts cost 12 dollarsî .

Student	6	d	12
Anna*	amount of doughnuts	doughnuts	cost
Ben	numeral	pronumeral	numeral
Cath	amount of doughnuts	cost per doughnut	Total price
Dan	six	doughnuts	cost $12
Ellie	number of doughnuts	doughnuts	price

* by far most common response

Slide 10: Interpreting student work 6d = 12

Interpreting student work 6d=12

ì 6 doughnuts cost 12 dollarsî

Only one correct!

Student	6	d	12
Anna*	amount of doughnuts	doughnuts	cost
Ben	numeral	pronumeral	numeral
Cath	amount of doughnuts	cost per doughnut	Total price
Dan	six	doughnuts	cost $12
Ellie	number of doughnuts	doughnuts	price

* by far most common response

Slide 11: Interpreting student work 6d = 12

Interpreting student work 6d=12

ì 6 doughnuts cost 12 dollarsî

Fran wrote this incorrect equation

$2\ d = 12$

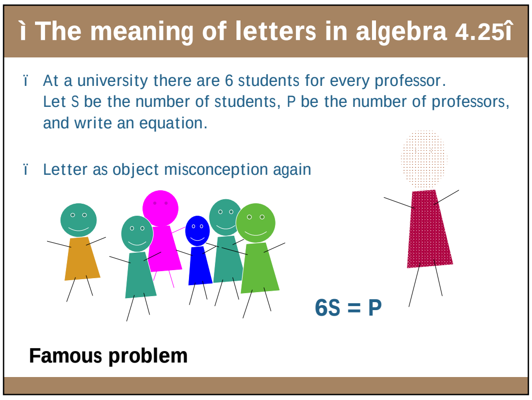
2 = cost of each doughnut

d = number of doughnuts

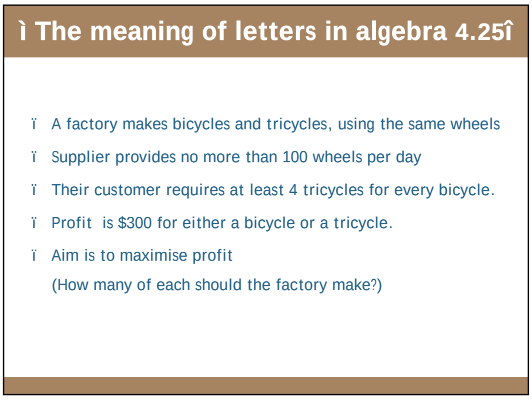
12 = overall cost

Unusual incorrect response, but Fran is one of the few students who thought carefully about what she really meant.

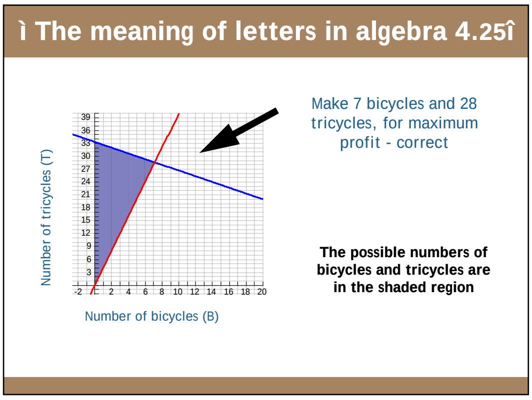
Slide 12: Interpreting student work 6d = 12



Slide 13: The meaning of letters in algebra: 4.25 — famous problem



Slide 14: The meaning of letters in algebra: 4.25



Slide 15: The meaning of letters in algebra: 4.25 — bicycle problem

A famous problem

The question posed on slide 13 is a very famous problem that shook the mathematics education world in the late 1970s when it was discovered that even apparently successful students in high year levels made many errors on this question. Ask participants to write the equation which responds to this problem:

> 'At a university there are 6 students for every professor. Let S be the number of students, P be the number of professors'.

The visual image (shown on the slide or mentally constructed) leads to the equation $6S = P$ when we think of S standing for students instead of the correct $S = 6P$, where S, the algebraic letter, stands for the number of students.

For many people the writing of the correct equation ($S = 6P$) seems counter-intuitive. Participants could be reminded that it is best to think that S represents the number of students (it is not a student).

A strategy to check the accuracy of the equation is for participants to try and substitute numbers into the equations. Try $S = 2$, $S = 5$, ... and then establish whether the solution is valid.

This activity is developed from a teaching strategy found on the Mathematics Continuum P–10:

Indicator of progress: The meaning of letters in algebra: 4.25 – Activity 3: Confusions about the meanings of letters (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/structure/St42504P.htm#a3)

Linear programming

Use slides 14–18: bicycle problem

Slide 14 refers to a linear programming problem which will illustrate the 'letter as object' misconception in senior mathematics students which occurs very often. The link with VCE Mathematics is important; seemingly basic misconceptions cause problems with advanced mathematics.

Invite participants to write the equations to the following problem:

- A factory makes bicycles and tricycles, using the same wheels.
- Supplier provides no more than 100 wheels per day.
- Their customer requires at least 4 tricycles for every bicycle.
- Profit is $\$300$ for either a bicycle or a tricycle.
- Aim is to maximise profit – how many of each should the factory make?

Correct graphical solution

Slide 15 illustrates a correct graphical solution. The point is that getting it right depends on the correct algebraic formulation. (It is acknowledged that this isn't a particularly meaningful realistic problem; the point is about algebraic formulation, not linear programming.)

Slide 16 explains the solution to the linear planning program. The most common errors in linear programming are in setting up the equations. Reversal, as shown in red, is very common (30% of uni students make this mistake).

The following strategies may help students write equations.

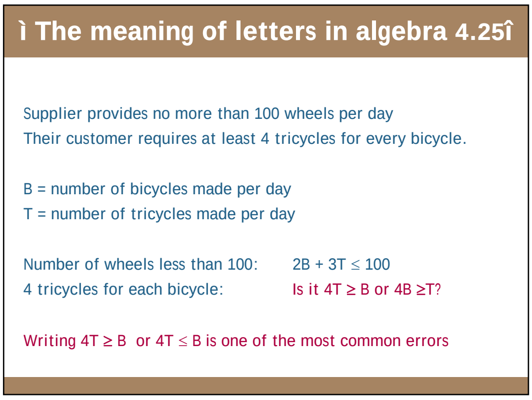
- Accept you are likely to make a mistake.
- Write the equation down.
- Test it with numbers.

For example in relation to the tricycle problem:

If there are 4 tricycles for every bicycle then:

- Let T = number of tricycles
- Let B = number of bicycles
- Try $T = 4B$. If $T = 400$, then $B = 100$ which supports the original problem.

The next slide shows the right and wrong solution graphically – just for closure.



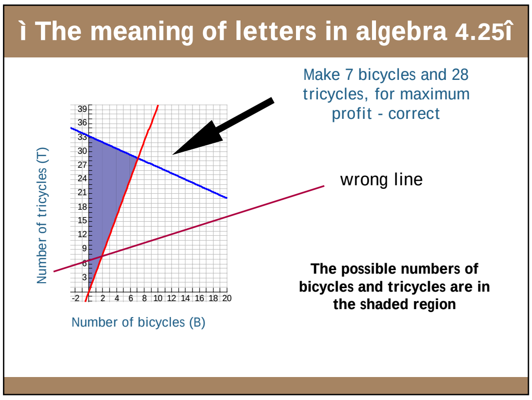
Slide 16: The meaning of letters in algebra: 4.25

Graphical solution to the tricycle problem

Slide 17 uses a graph to show participants how the graphical linear programming solution is affected by the wrong equation.

The correct red line is $T = 4B$

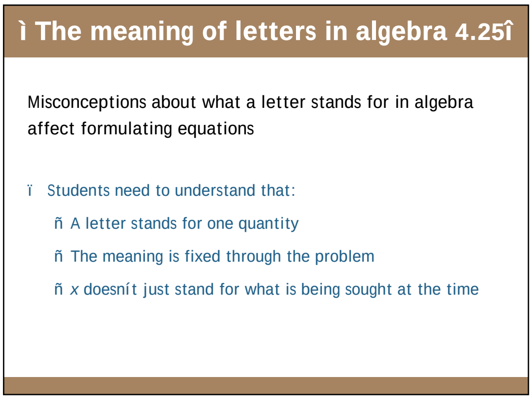
The incorrect maroon line is $4T = B$. Too many students do this and it is a consequence of not thinking of the letter as a number.



Slide 17: The meaning of letters in algebra: 4.25

Misconceptions about what a letter stands for in algebra affect formulating equations

Slide 18 describes how misconceptions also affect formulating equations. Remind participants that students need to understand that the letter stands for one quantity and that the letter's value is fixed through the problem so that 'x' doesn't just stand for what is being sought at the time.



Slide 18: The meaning of letters in algebra: 4.25

ì The meaning of letters in algebra 4.25î

Illustration 2- Diagnostic item

Write an equation that describes the following situation.
Use b to stand for the number of blue pencils and r to stand for the number of red pencils.

I bought some red pencils and some blue pencils and spent a total of 90 cents. The blue pencils cost 10 cents each and the red pencils cost 6 cents each.

Slide 19: The meaning of letters in algebra: 4.25 — Illustration 2

Explain the following student answers

ï $10b + 6r = 90$ (correct)
ï $b+r = 90$
ï $6b+5r = 90$
ï $b=3,\ r = 10$

I bought some red pencils and some blue pencils and spent a total of 90 cents. The blue pencils cost 10 cents each and the red pencils cost 6 cents each. Write an equation to describe this situation.

Slide 20: Explain the following student answers

Formulating equations

Slide 19 illustrates how students' misconceptions about letter as an object can impact their writing of equations.

Use slides 19–20: Pencil problem

Invite the participants to write a correct and incorrect equation that describes the following situation:

- I bought some red pencils and some blue pencils and spent a total of 90 cents. The blue pencils cost 10 cents each and the red pencils cost 6 cents each.

Use 'b' to stand for the number of blue pencils and 'r' to stand for the number of red pencils. This question was taken from illustration 2 described in the Mathematics Continuum P–10:

- Indicator of progress: The meaning of letters in algebra: 4.25 – Illustration 2 (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/structure/St42504P.htm)

Student responses to the pencil problem

Slide 20 provides some student responses.

Invite the participants to try to work out what the students are thinking with these responses.

The following responses have been found typically from research – these responses are all reasonably common in Australia and elsewhere.

- $10b + 6r = 90$ (correct)
- $b + r = 90$

  The red and the blue pencils cost 90 cents – interpreting b as the blue pencils, and r as red pencils.
- $6b + 5r = 90$

  Student has arithmetically found possible numbers of pencils, and then is writing 6 blue pencils + 5 red pencils cost 90 cents (which is true – student does not understand that algebra provides a different way of solving problems compared with arithmetically) .
- $b = 3,\ r = 10$

  Student has found a solution but not written an equation (and probably didn't do it by algebra either!).

Further information

Teachers should try to identify student misconceptions by observing:

- Evidence of 'letter as object' – in this case: b = blue pencils.
- Students who think that they have to solve problems before they write equations. They don't appreciate that they can write equations in order to solve problems (see next section).

- Obstacles with literacy. Sometimes there is a claim that literacy is the obstacle to solving worded problems, but this is not the case. Teachers think it is literacy, but research shows that it may well be the maths. This is shown if the same problem is given with easy numbers (95% correct) and then change to a letter or a hard number (the facility drops considerably).

There are further illustrations of this misconception noted on Mathematics Continuum P–10:

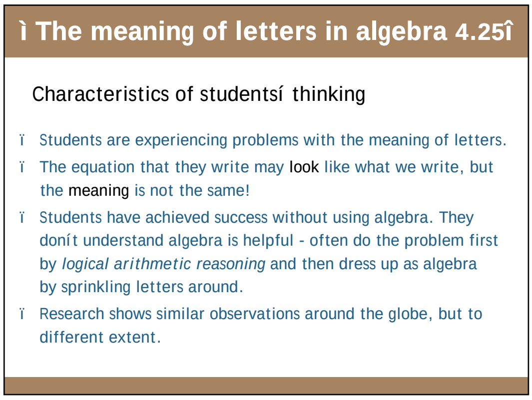
- Indicator of progress: The meaning of letters in algebra: 4.25 (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/structure/St42504P.htm)

Research has shown that using other pronumerals such as x, y, z are not any better than letters related to the problem such as: t = number of hours etc.

Characteristics of students' thinking about algebra

Slide 21 describes some key messages which characterises student thinking. These messages include:

- Students are experiencing problems with the meaning of letters.
- The equations that students write may look like what we write, but the meaning is not the same.
- Students will try to achieve success without using algebra. They don't understand algebra is helpful – they often do the problem first *by logical arithmetic reasoning* and then dress it up as algebra by sprinkling letters around.
- Research shows similar observations around the globe.

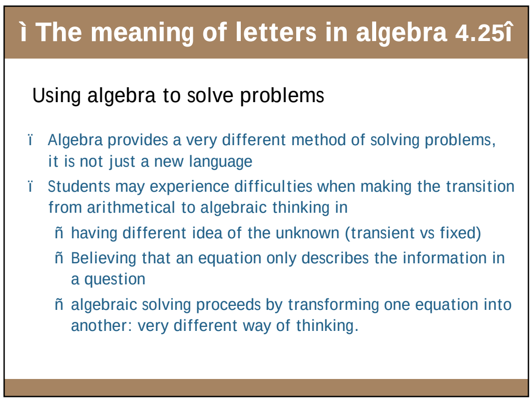


Slide 21: The meaning of letters in algebra: 4.25 — Characteristics of students' thinking

Using algebra to solve problems

Slide 22 provides information about how students can use algebra to solve problems. The important issue is that solving problems using algebraic equations is logically very different from solving arithmetical problems.

- Sometimes students solve the problem then dress it up to look like algebra, without actually using algebra.
- Students need to realise that the unknown has only one value, and that does not change with the solution of the problem.
- An equation is a symbolic representation rather than a shorthand description of the problem.
- The process of solving a problem using algebraic methods requires a whole new way of thinking. It requires them to understand the meaning of the equivalence which is represented now by the equal sign.



Slide 22: The meaning of letters in algebra: 4.25 — Using algebra to solve problems

ì The meaning of letters in algebra 4.25î

Diagnosing studentsí thinking

MARK AND JAN

Mark and Jan share $47, but Mark gets $5 more than Jan. How much do they each get?

Slide 23: The meaning of letters in algebra: 4.25 — Diagnosing students' thinking

Diagnosing students' thinking

Slide 23 introduces a new activity which shows that students may change the value of the pronumeral during the solution of a problem. This activity, using slides 23–33, provides examples of how teachers can diagnose student thinking. It is adapted from a teaching strategy found on the Mathematics Continuum P–10.

The concept of an unknown but fixed number, or of a variable, is difficult to convey in words. When students use algebra to solve problems, it is important that they write down clearly at the start of a problem what quantity each variable is standing for. In any one problem, one letter is allocated to one unknown quantity.

This is the type of error that students make in their subsequent use of algebra when they do not appreciate this fundamental principle. Teachers need to emphasise this principle from the start of algebra and frequently revisit it in subsequent years.

Use slides 23–33: Diagnosing student thinking

Diagnosing student thinking

This activity will illustrate how teachers can diagnose students' thinking, through questioning student responses to algebraic problems.

Teachers are encouraged to provide problems which will require students to use algebra, and they cannot just solve mentally. The risk is that otherwise some students just decorate their solution with algebra. This form of decoration is illustrated also in the following examples.

This activity is developed from a teaching strategy found on the Mathematics Continuum P–10:

- Indicator of progress: The meaning of letters in algebra: 4.25 (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/structure/St42504P.htm)

The problem

Slide 23 states the following problem:

- Mark and Jan share *$47*, but Mark gets *$5* more than Jan. How much do they each get?

Ask participants to think of two or more ways to do this problem.

Suggested correct solutions:

- Using algebra in several variations

  e.g. $x + (x + 5) = 47$,

  $x + (x - 5) = 47$,
- Guess and check
- Logical arithmetical reasoning in two different ways:

  Give Mark the *$5* first, leaving *$42*, which is then shared between them – Jan gets *$21* and Mark gets *$21* + *$5*.

Teachers need to be aware that some students do not understand that algebra **does the work for you** – they may use arithmetic methods and then dress up their solution as algebra.

Brenda and Wylie's responses

Slide 24 provides examples of student solutions including Brenda and Wylie.

Brenda

$47/2 = 23.5 - 2.5 = x$

$47/2 = 23.5 + 2.5 = y$

Wylie

$y = (47 - 5) / 2 + 5 = 42/2 + 5 = 26$

$x = (47 - 5) / 2 = 42/2 = 21$

Other students

$y = (T - D) / 2 + \mathrm{D}$, $x = (T - D) / 2$

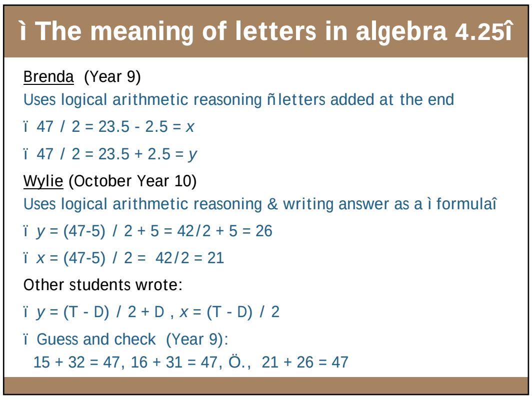
Bring participants attention to:

- How Brenda is using logical arithmetic reasoning – letters were added at the end.
- How Wylie is using logical arithmetic reasoning & writing answer as a 'formula'.
- How Brenda and Wylie are doing all the work of solving the problem **before** they turn to algebra.
- How both of them have not yet found algebra a powerful problem solving method .How they are working across the page with sometimes inappropriate = = =. This is not the focus of discussion but is worth noting.
- Wylie in Year 10 is still 'dressing up' with algebra and doesn't understand that the algebra does the work for him.
- Other students solved the problem by using the symbols T and D.

  T = total

  D = difference
- Some students even used the guess and check method given by the example:

$15 + 32 = 47,\ 16 + 31 = 47,\ \ldots.,\ 21 + 26 = 47$



Slide 24: The meaning of letters in algebra: 4.25 — Brenda and Wylie's responses

Wylie (Year 11)

Slide 25 shows Wylie's response to the same question.

Wylie in the following year had changed his thinking. He has made good progress – many students don't. He has made the transition to algebra (but he has not finished the problem – he has not checked that he has answered the question set).

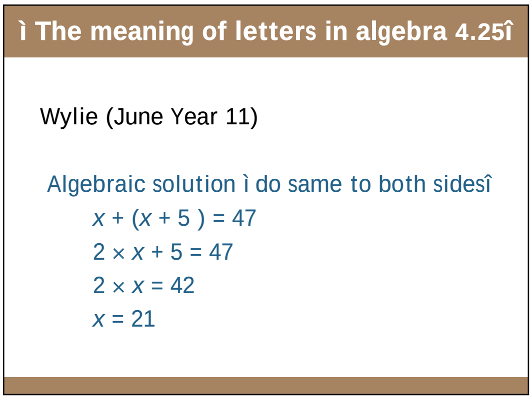
Wylie was using the algebraic approach by Year 11 as shown by the algebraic solution 'do same to both sides':

$x + (x + 5\ ) = 47$

$2 \times x + 5 = 47$

$2 \times x = 42$

$x = 21$



Slide 25: The meaning of letters in algebra: 4.25 — Wylie (Year 11)

ì The meaning of letters in algebra 4.25î

How has Les used x?

Les begins by writing **$5 + x = 47$**

L: x is what is left out of $47 if you take 5 off it.

I: What might the x be?

L: Say she gets $22 and he gets $27. They are sharing two $x$ís.

I: What are the two $x$ís?

L: UnknownsÖthey are two different numbers, 22 and 27.

I: So what is this x? (pointing to $5 + x = 47$)

L: That was what was left over from $47, so its $42.

Slide 26: The meaning of letters in algebra: 4.25 — Les's thinking

Les's thinking

Slide 26 describes a conversation between Les and the interviewer which illustrates the type of questioning used to interrogate students and provide an insight into their thinking.

Les begins by writing:

[Les:]	$5 + x = 47$
[Les:]	x is what is left out of *$47* if you take *5* off it.
[Interviewer:]	What might the x be?
[Les:]	Say she gets *$22* and he gets *$27*. They are sharing two x's.
[Interviewer:]	What are the two x's?
[Les:]	Unknowns. They are two different numbers, *22* and *27*.
[Interviewer:]	So what is this x? (pointing to $5 + x = 47$)
[Les:]	That was what was left over from *$47*, so its *$42*.

ì The meaning of letters in algebra 4.25î

Les refers to x as meaning several different things, he informally tracks thinking with ëalgebraí

Les begins by writing $5 + x = 47$

L: x is what is left out of $47 if you take 5 off it.
I: What might the x be?
L: Say she gets $22 and he gets $27. They are sharing two $x$ís.
I: What are the two $x$ís?
L: UnknownsÖthey are two different numbers, 22 and 27.
I: So what is this x? (pointing to $5 + x = 47$)
L: That was what was left over from $47, so its $42.
(3 different meanings for x simultaneously)

Slide 27: The meaning of letters in algebra: 4.25 — Tracking Les's thinking

Tracking Les's thinking

Slide 27 tracks Les's thinking.

Use this information to help participants to answer the question:

- How has Les used x?

Suggested responses from participants may include:

- Les is informally tracking his thinking.
- He is using arithmetic reasoning where he is working out a sequence of unknowns that keep changing.
- Les hasn't made the change to thinking about what an unknown in arithmetic means to what it means in algebra. In algebra, an unknown has a fixed value for that problem.

ì The meaning of letters in algebra 4.25î

How has Joel used x?

ï Joel writes x (for Janís amount)

ï Then writes $x + 5$ (for Mark)

ï Then $x + 5 = 47$

ï I: Points to $x + 5 = 47$. What does this say?

ï J: (itís) the amount they both get. The amount that Jan gets. I just like to keep the three of them, 47 dollars, x and 5 dollars and make something out of them.

Slide 28: The meaning of letters in algebra: 4.25 — Joel's thinking

Joel's thinking

Slide 28 shows the discussion between the interviewer and Joel

Ask the participants to refer to the following example and answer the question:

- How has Joel used x?"

Joel writes	x	(for Jan's amount)
Then writes	$x + 5$	(for Mark)
Then	$x + 5 = 47$	
[Interviewer:]	Points to $x + 5 = 47$. What does this say?	
[Joel:]	(it's) the amount they both get. The amount that Jan gets. I just like to keep the three of them, *47* dollars, x and *5* dollars, and make something out of them.	

Tracking Joel's thinking

Slide 29 shows that Joel provides for multiple ways of representing 'x'.

Explain to participants that Joel's reasoning provides for multiple ways of representing 'x'.

Joel writes	x	(for Jan's amount)
Then writes	$x + 5$	for Mark
Then	$x + 5 = 47$, where x is 'the amount they both get' ($42) and as well as Jan's amount.	

Tim's thinking

Slide 30 shows the discussion between the interviewer and Tim about his solution to the problem.

Ask participants whether they can see how Tim has used 'x'?

Tim writes	$x + 5$	for Mark's amount
Then writes	$x = x + 5$	saying the x after the equal sign is 'Jan's x'

Slide 31 demonstrates Tim's use of x as a general label for all unknown quantities

[Tim:]	(pointing to first x in $x + 5 = x$) That's Mark's x.
[Interviewer:]	And why do we add 5 to it?
[Tim:]	Because Mark has 5 more dollars than Jan. No, that's not right; it should be Jan's x plus 5 equals Mark's x.
[Interviewer:]	Could you write an equation to say that Mark and Jan have $47 in total? You don't have to work out the answer first.
[Tim:]	x divided by a half equals x (writes $x \div \frac{1}{2} = x$)

Participants may point out that Tim is using x as a general label for **all** unknown quantities. Jan's x, Mark's x and probably also $x = 42$ (last line).

ì The meaning of letters in algebra 4.25î

Joel: multiple and shifting referents for x

- Joel writes *x* (for Janís amount)
- Then writes *x + 5 for Mark*
- *Then x + 5 = 47*
- I: Points to $x + 5 = 47$. What does this say?
- J: (itís) the amount they both get. The amount that Jan gets. I just like to keep the three of them, 47 dollars, *x* and 5 dollars and make something out of them.
- *x* as ì the amount they both getî ($42) and as well as Janís amount

Slide 29: The meaning of letters in algebra: 4.25 — Tracking Joel's thinking

ì The meaning of letters in algebra 4.25î

How has Tim used x?

Tim writes *x* + 5 for Markís amount
Then writes *x* = *x* + 5, saying the *x* after the equal sign is ì Janís *x*î

T: (Pointing to first *x* in *x*+5 = *x*) Thatís Markís *x*.
I: And why do we add 5 to it?
T: Because Mark has 5 more dollars than Jan. No, thatís not right, it should be Janís *x* plus 5 equals Markís *x*.
I: Could you write an equation to say that Mark and Jan have $47 in total ? You donít have to work out the answer first.
T: *x* divided by a half equals *x* (writes *x* ÷ 1/2 = *x*)

Slide 30: The meaning of letters in algebra: 4.25 — Tim's thinking

ì The meaning of letters in algebra 4.25î

Tim: uses x as a general label for all unknown quantities.

Tim writes *x* + 5 for Markís amount
Then writes *x* = *x* + 5, saying the *x* after the equal sign is ì Janís *x*î

T: (pointing to first *x* in *x*+5 = *x*) Thatís Markís *x*.
I: And why do we add 5 to it?
T: Because Mark has 5 more dollars than Jan. No, thatís not right, it should be Janís *x* plus 5 equals Markís *x*.
I: Could you write an equation to say that Mark and Jan have $47 in total ? You donít have to work out the answer first.
T: *x* divided by a half equals *x*
(writes *x* ÷ 1/2 = *x*)

Slide 31: The meaning of letters in algebra: 4.25

ì The meaning of letters in algebra 4.25î

Can you see what is bothering Leonie?

Leonie writes

(x + 5) + y = 47, and cannot progress beyond this point Why?

Leonie explains that

ï (x+5) is the money that Mark has

ï this says it is $5 more than Janís money

ï y is the money that Jan has

ï Leonie believes that the equation

(x+5) + x = 47 is wrong. Why?

Slide 32: The meaning of letters in algebra: 4.25 — What is worrying Leonie?

ì The meaning of letters in algebra 4.25î

Explanation of Leonieís thinking

ï Leonie knows that the numerical amounts of x and y are the same, but she uses the separate letters because she believes Mark have physically different money.

ï For Leonie, x represents the actual money, not the amount of money.

ï Another instance of ì letter as objectî evident in studentís work long after introductory lessons on algebra.

Slide 33: The meaning of letters in algebra: 4.25 — Explanation of Leonie's thinking

ì The meaning of letters in algebra 4.25î

Summary

ï Uncertainties and misconceptions about the meanings of letters lie behind many difficulties with algebra

ñ writing expressions

ñ formulating equations.

ï Examine studentsí work closely to identify their difficulties, and then address them.

ï Use teaching strategies that emphasise that algebraic letters stand for numbers, and that there is a specific meaning for a letter throughout one problem.

Slide 34: The meaning of letters in algebra: 4.25 — Summary

What is worrying Leonie?

Slide 32 demonstrates Leonie's thinking.

Ask participants if they can see what is worrying Leonie.

Refer to the following example:

- Leonie writes $(x + 5) + y = 47$, and cannot progress beyond this point.
- Leonie explains that:
  - $(x+5)$ is the money that Mark has
  - this says it is *$5* more than Jan's money
  - y is the money that Jan has.
- Leonie believes that the equation
  - $(x+5) + x = 47$ is wrong.

Explanation of Leonie's thinking

Slide 33 explains Leonie's thinking

Leonie knows that the numerical amounts of x and y are the same, but she uses the separate letters because she believes Mark and Jan have physically different money.

For Leonie, x represents the actual money, not the amount of money.

This example illustrates another instance of 'letter as object' evident in students' work long after introductory lessons on algebra.

Summary of the diagnosing student thinking activity

Slide 34 provides a summary of this activity.

Uncertainties and misconceptions about the meanings of letters lie behind many difficulties with algebra. Many of these misconceptions arise from the use of letters in our everyday world. (Note that there are conflicts with other notations such as in programming and spreadsheet algebra, other symbolic languages and perhaps even text on a phone.)

These misconceptions manifest themselves in terms of:

- writing expressions
- formulating equations.

Teachers need to examine students' work and question students in order to identify their difficulties, and then address them by using teaching strategies that emphasise that:

- algebraic letters stand for numbers
- there is a specific meaning for a letter throughout one problem.

How many letters in my name?

Slide 35 and 36 introduce a teaching strategy which emphasises that algebraic letters (pronumerals) stand for numbers and that equations are true for some numerical values of the pronumerals and not for others.

This activity is developed from a teaching strategy found on the Mathematics Continuum P–10:

Indicator of progress: The meaning of letters in algebra: 4.25 – Activity 2: How many letters in my name? (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/structure/St42504P.htm#a2).

This activity provides an opportunity for participants to build simple equations valid for their own names.

Inform the participants of how this activity can be used in a classroom context. Students create an equation of their name and they have this equation checked by another student who then forwards the equations on slips of paper to the teacher who writes them unidentified onto the board.

Once all the equations are collected, students work out which equation belongs to which person.

How many letters in your name?

Use slides 35–39

Slide 36 asks the participants to create equations for their names.

Ask participants to:

- Make up three equations for their name.
- Try to include the variety of equations which students might write (correct and incorrect).
- Pool their equations and think about what different equations will reveal about students' thinking.

Three different examples for each of four students

Slide 37 provides examples of three different equations for each of four students. This animated slide illustrates errors that students may make when constructing equations.

In class, teachers should be aware that:

- Some errors will get through the peer checking – the teacher needs to be alert to avoid this.
- Need to write up students' names so everyone knows how to spell them and identify how many letters.
- Better with a small group of names than the whole class (too much to do).
- Use an electronic whiteboard to help students to visualise their thinking.

In class discussion:

- Students should be able to identify both correct and incorrect equations.
- Later, given another set of equations, they have to decide which equation belongs to which person.

ì The meaning of letters in algebra 4.25î

How many letters in my Name?

Let a = the *number of letters* in my first name.
Let b = the *number of letters* in my family name.

For **Lini Marandri**, $a = 4$, $b = 8$
Sample equation: $a + b$
$= 4 + 8$
$= 12$

For **Thy Vo** $a = 3$, $b = 2$
Sample equation: $a + b$
$= 3 + 2$
$= 5$

Slide 35: The meaning of letters in algebra: 4.25 — How many letters in my name?

ì The meaning of letters in algebra 4.25î

Try it at your table:

ï Make up 3 equations for your name
ï Try to include the variety of equations which students might write (correct and incorrect)
ï Pool your equations and think about what different equations will reveal about studentsí thinking.

Slide 36: The meaning of letters in algebra: 4.25

Sample Equations

Some will be identities ñtrue for everyone!

Probably a bracketing error b ñ(a+1) = 0

Place value confusion is common with beginners

Lini Marandri, a=4, b = 8	4+a = [illegible]	a+a = b	ba=84
Thy Vo, $a = 3$, $b = 2$	a+b = 5	a÷b = 1.5	a+a=a ×2
Robert Menzies, a=6, b = 7	b-a=1	a-a = b-b	b-a+1 = 0
John Curtin, a=4, b=6	a×b = 24	2 ×a + b + a + 1 = 19	

Slide 37: Sample Equations

Sample Equations

No need to stay with linear equations

Lini Marandri, a=4, b = 8	4+a = b	a[illegible] = b	b[illegible]34
Thy Vo, a = 3, b = 2	a+b = 5	a÷b [illegible]	a+a=a ×2
Robert Menzies, a=6, b = 7	b-a=1	a-a = b-b	b-a+1 = 0
John Curtin, a=4, b=6	a×b = 24	2 ×a + b + a + 1 = 19	

These equations can be easily solved by guess-check, because there are only a few numbers to try.

Slide 38: Sample Equations

Discuss the following with participants using the animated slide:

- Letters generated from the names stand for numbers.
- Some equations hold for everyone, others only for a few people or one person.
- Errors with brackets.
- Syntax errors and other errors e.g. $ba = 84$. We mean ba to mean b multiplied by a. However the student, Lini, has just placed the numbers side by side. Lini may also be confused about place value where the 8 is written in the tens place.
- Some of the equations are identity laws. An example of an identity will be true regardless of the values of the pronumerals. Two identities are highlighted by arrows. They are $a - a = b - b$, and $a + a = a \times 2$. This is an important point to raise with students.

Non-linear equations

Slide 38 shows two equations that have been generated by students which are not linear equations. These students have multiplied and divided their pronumerals which is a legitimate option. The class may be able to solve these equations by the guess-check method.

ì The meaning of letters in algebra 4.25î

Main ideas: how many letters in my name

- ï Letter stands for number ñunknown to audience ñpossibly can be found by audience
- ï Reinforces simple substituting, basic syntax, etc
- ï Students may make harder equation than teacher expects ñcreativity, diversity
- ï Some are equations and some identities: some equations can belong to one person, some to more than one person, and some to everyone
- ï Equation solving by guess-check-improve.

Slide 39: The meaning of letters in algebra: 4.25 — Summary of the name equation activity

Summary of the name equation activity

Slide 39 provides an overall summary illustrating the aims of the 'How many letters in my name' activity. This activity reinforces the notion that a letter stands for a number – unknown to the audience – which possibly can be found by the audience. It also:

- reinforces simple substituting, basic syntax, etc
- gives the students an opportunity to make harder equations than teacher expects, thus providing for creativity and diversity of responses
- illustrates the concept of equations and identities, showing that some equations can belong to one person, some to more than one person, and some to everyone
- provides a means of solving equations by using guess-check-improve.

Assessment

- ï Assessment practices are an integral part of teaching and learning (PoLT Principle 5)
- ï Teachers are encouraged to use evidence from assessment to inform planning and teaching.
- ï These Continuum items described in the module are intended as diagnostic assessment.
- ï Teachers should aim to understand what their students are trying t[o] when they try to write algebra. Teachers can then design instructi[on that] makes sense to students and hence changes their thinking more effectively.

Slide 40: Assessment

Assessment

Slide 40 makes some concluding statements about Module 5 by linking back to assessment. Remind participants that teachers should use evidence from assessment to inform planning and teaching. This is reinforced by principle 5 of Principles of Learning and Teaching P–12.

See:

- resource 1
- The Principles of Learning and Teaching P-12 Unpacked (http://www.education.vic.gov.au/studentlearning/teachingprinciples/principles/unpacked.htm)

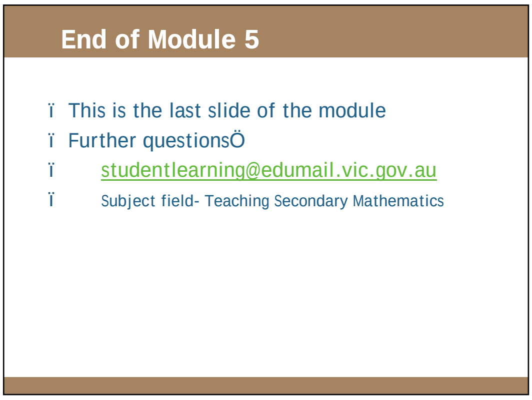
Conclusion

Teachers who understand what their students are trying to say when they try to write algebra can then design activities which makes sense to students and hence change the students' thinking more effectively.

Slide 41 is the concluding slide to Module 5 'Understanding students' mathematical thinking: Focus on algebra and the meaning of letters'.

There are 8 more professional learning modules:

1. Overview of learning in the Mathematics Domain
2. Overview of the Mathematics Developmental Continuum P–10
3. Narrowing the achievement gap: Focus on fractions
4. Conducting practical and collaborative work: Focus on contours
6. Using a range of strategies and resources: Focus on percentages
7. Learning through investigation: Focus on chance and variability
8. Working mathematically: Focus on a range of challenging problems
9. Conclusion: Planning for improvement in mathematics



Slide 41: End of Module 5

Resource 1: Principles of Learning and Teaching P–12 and their components

- Principles of Learning and Teaching P–12 and their components (http://www.education.vic.gov.au/studentlearning/teachingprinciples/principles/principlesandcomponents.htm)

Students learn best when:

The learning environment is supportive and productive. In learning environments that reflect this principle the teacher:

1.1) builds positive relationships through knowing and valuing each student

1.2) promotes a culture of value and respect for individuals and their communities

1.3) uses strategies that promote students' self-confidence and willingness to take risks with their learning

1.4) ensures each student experiences success through structured support, the valuing of effort, and recognition of their work.

The learning environment promotes independence, interdependence and self motivation. In learning environments that reflect this principle the teacher:

2.1) encourages and supports students to take responsibility for their learning

2.2) uses strategies that build skills of productive collaboration.

Students' needs, backgrounds, perspectives and interests are reflected in the learning program. In learning environments that reflect this principle the teacher:

3.1) uses strategies that are flexible and responsive to the values, needs and interests of individual students

3.2) uses a range of strategies that support the different ways of thinking and learning

3.3) builds on students' prior experiences, knowledge and skills

3.4) capitalises on students' experience of a technology rich world.

Students are challenged and supported to develop deep levels of thinking and application. In learning environments that reflect this principle the teacher:

4.1) plans sequences to promote sustained learning that builds over time and emphasises connections between ideas

4.2) promotes substantive discussion of ideas

4.3) emphasises the quality of learning with high expectations of achievement

4.4) uses strategies that challenge and support students to question and reflect

4.5) uses strategies to develop investigating and problem solving skills

4.6) uses strategies to foster imagination and creativity.

Assessment practices are an integral part of teaching and learning. In learning environments that reflect this principle the teacher:

5.1) designs assessment practices that reflect the full range of learning program objectives

5.2) ensures that students receive frequent constructive feedback that supports further learning

5.3) makes assessment criteria explicit

5.4) uses assessment practices that encourage reflection and self assessment

5.5) uses evidence from assessment to inform planning and teaching.

Learning connects strongly with communities and practice beyond the classroom. In learning environments that reflect this principle the teacher:

6.1) supports students to engage with contemporary knowledge and practice

6.2) plans for students to interact with local and broader communities and community practices

6.3) uses technologies in ways that reflect professional and community practices.