DEVELOPING THE 'BIG' IDEAS IN NUMBER

One of the main aims of school mathematics is to create mental objects in the mind's eye of children which can be manipulated flexibly with understanding and confidence. A prolonged reliance on inefficient strategies such as "make-all-count-all" or "counting-by-ones" is both developmentally dangerous and professionally irresponsible..

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"... NUMBER SENSE refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful and efficient strategies for managing numerical situations.

It results in a view of numbers as meaningful entities and the expectation mathematical manipulations and outcomes should make sense.

Those who use mathematics in this way continually utilise a variety of internal "checks and balances" to judge the reasonableness of numerical outcomes" (p3).

McIntosh, A., et al (1997) *Number Sense in School Mathematics - Student Performance in Four Countries*, Perth: MASTEC

DEVELOPING A SENSE OF NUMBER

What is needed?

An understanding of number and operations together with an ability and ***inclination***

- to use this in ***flexible*** ways to make mathematical judgments
- to develop useful ***strategies*** for handling numbers
- to communicate, process and ***interpret*** information."

McIntosh, A., Reys, B. & Reys, J. (1992) A proposed framework for examining basic number sense, *For the Learning of Mathematics*, 12(3), 2-8

Three essential underpinnings:

(i) NUMERATION (PLACE-VALUE) An understanding of numbers and an ability to think of them in more than one way (rename numbers).

(ii) MEANING FOR THE OPERATIONS (CONCEPTS) An understanding of what the operations do, an ability to recognise the operation symbols and an ability to write and interpret symbolic statements.

(iii) MENTAL STRATEGIES (NUMBER FACTS) A working knowledge of addition and subtraction facts to 20 and multiplication and division facts to 100, based on efficient non-counting, mental strategies.

Reference: Booker, G., Bond, D., Briggs, J. & Davey, G. (1997) *Teaching Primary Mathematics* (2nd Edition), Longman Cheshire: Melbourne

Four essential aspects:

- FOCUS ON SPEAKING Encourage children to invent and present action stories, count aloud, explain their solutions, model and acknowledge more efficient strategies. Model correct language usage, encourage the use of appropriate terminology.

- FOCUS ON LISTENING Model more efficient counting strategies, count on from larger, skip counting (small numbers only), count to support place-value, eg, "onety-two, onety-three …twoty-one, twoty-two, …threety-five, threety-six…" Listen carefully to student explanations, re-state to value contributions, use to direct teaching

- FOCUS ON READING 'Read' physical collections, models, displays, adopt a "say what you see" approach, practice subitisation, ie, say how many without counting, teach children to read numbers, eg, 367 can be read to the tens place as 36 tens and 7 ones, read very large numbers from newspapers/magazines

- FOCUS ON WRITING Explore informal writing. Delay initial recording until children have access to mental strategies, expect non-modelled solutions, encourage students to record their explanations, use diagrams and symbols appropriately.

RECOGNISE THAT EARLY NUMERATION INVOLVES MORE THAN COUNTING

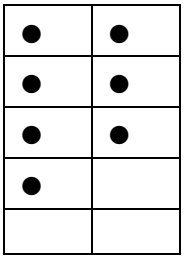
While the ability to count is important, children need to know much more before they are ready to use numbers flexibly, with understanding.

NUMERATION INVOLVES:
- **One-to-one correspondence**
- **Recognising that "three" means a collection of three whatever it looks like.**
- **Recognising that the last number counted represents the number in the collection**
- **Matching words and/or numerals to collections less than 10 (knowing the number naming sequence).**
- **Reading, writing, and using the words and numerals for the numbers 0 to 9.**

BUT MORE IMPORTANTLY IT INVOLVES:
- **Recognising collections of up to five objects without counting (subitise).**
- **Being able to name numbers in terms of their parts (part-part-whole).**
- **Trust the count (Willis, 2002)**

ESTABLISH PART-PART-WHOLE IDEAS FOR NUMBERS 0-10:
Using a Ten Frame, for example,

7 is
6 and 1 more, 1 and 6
1 less than 8
5 and 2, 2 and 5
double 3 and 1 more
3 and 4, 4 and 3
0 and 7



Use visual recognition activities involving collections, games and the use of *Ten-Frames*, for example,

- Make a permanent *Ten-Frame* on a white-board or felt board. Use with different numbers to review what is known and extend to new numbers.
- Make a frame for each number on a large poster, hang it in the classroom and add relationships as children discover them.
- Make a class book for each number based on part-part-whole ideas. Make a frieze for the classroom.
- Make a set of number cards for the numbers 0 to ten. One set each for numerals, words, collections, different ten frames, and part-part-whole relationships (5 different sets in all). Play collection games, 'Concentration', 'Snap' etc
- Have children sit at tables of ten arranged in pairs like a *Ten-Frame*. Ask: How many at the table?
- Send an A3 version of a *Ten-Frame* home to be used with fridge magnets. Encourage all members of the family to say what they see and why.
- Make a playground version of a *Ten-Frame* - have children jump in and out of the frame to make a given number in as many ways as they can.

ESTABLISH DOUBLES FACTS TO 20, EXPLORE MORE EFFICIENT STRATEGIES:

- Use 2 ten-frames, 2 rows of a bead frame, or a 'double-decker' bus model to develop and extend doubles knowledge, devise number stories, explore strategies.

INTRODUCE & CONSOLIDATE PLACE-VALUE IDEAS

INTRODUCE PLACE-VALUE:

- ***Introduce the 'new' unit* - that is, ten ones is 1 ten via bundling and counting tens, eg, "1 ten, 2 tens, 3 tens, 4 tens …"**
- ***Introduce the names for multiples of ten* - language only, no symbols**
- ***Make, Name* and *Record* numbers 20-99 using appropriate models - eg, "make 6 tens 3 ones", read and write "sixty-three", record using a place-value chart**
- ***Make, Name* and *Record* numbers 10-19, pointing out inconsistency in language (should be onety-seven, onety-eight etc)**
- ***Consolidate place value knowledge***

REINFORCE PLACE-VALUE THROUGH REGULAR ACTIVITY:

- ***Make, Name, Read* and *Record* numbers using appropriate models**
- ***Compare* two numbers - Which is bigger? Why? How do you know?**
- ***Order* and *Sequence* numbers**
- ***Count* forwards and backwards in place-value parts**
- ***RENAME* numbers in as many different ways as possible**

EXTEND PLACE-VALUE IDEAS

Introduce the **second place-value pattern**, ie, the repeated use of hundred, tens and ones, as in 356 million 356 thousand 356 (ones) - this helps with 5 and more digit numeration

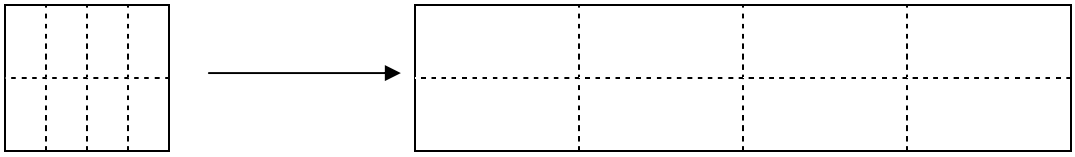
- Ensure the patterns related to adjacent place-value parts are understood and generalised, ie, "ten of these is one of those"… "1 tenth of these is one of those"

DEVELOP FRACTION IDEAS FROM INFORMAL, PRACTICAL EVERYDAY EXPERIENCE

- Review fraction language and ideas using discrete and continuous materials, record parts using written language, eg, 3 fifths

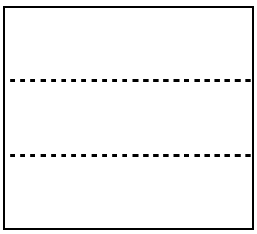
Continuous	*Discrete*
eg, 2 and 3 quarter pizzas	*eg, half the grade to art, half to the library*
eg. 2 thirds of the netball court	*eg. 2 out of 12 eggs are cracked*

- Evaluate foundation ideas, ie, recognition that equal parts required, the number of equal parts names the parts (denominator idea), and that the number of parts required tells how many (numerator idea)
- Involve children in **partitioning** – start with making and naming simple, mixed common fractions using **'halving strategy'** and paper folding. Generalise to diagrams derived from paper folding techniques, for example,

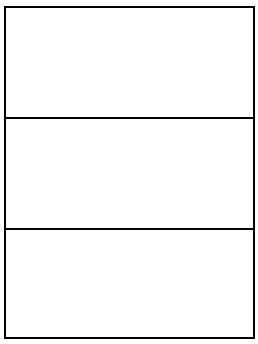


Paper Model Diagram derived from paper folding

- Extend partitioning strategies to **'thirding'** and **'fifthing'** derived from paper folding experiments and estimation based on reasoning about the size of the parts. Introduce the fraction symbol (3 'out of' 5).



Think: 3 parts must be
smaller than 2 parts
1 third less than 1 half
estimate
halve remaining part



'bank fold' - three equal parts derived from paper-folding → derived diagram

- Use Kindergarten Squares (coloured paper), plasticene, paper streamers to make and name fractions and extended range of fractions
- Introduce tenths via diagrams, make and name ones and tenths, introduce decimal recording as a new place-value part

- Extend partitioning techniques to develop understanding that thirds by fourths produce twelfths, tenths by tenths give hundredths
- Explore fraction renaming (equivalent fractions) using paper-folding, diagrams and fraction kits (where appropriate)
- Extend decimal fraction knowledge to hundredths and thousandths using diagrams, appropriate materials (eg, Linear Arithmetic Rods) and metric relationships, introduce percentage

INTRODUCE THE FOUR OPERATIONS IN A SYSTEMATIC WAY (build from the physical to the verbal to the symbolic)

CONCEPTS through action stories, modelling, 'make-all-count-all' strategies
- addition: *join, combine*
- subtraction: *take-away, missing addend, difference*
- multiplication: *groups of, arrays, regions, area, Cartesian product*
- division: *quotition, partition*

MENTAL STRATEGIES (basic facts) via visual imagery, part-part-whole ideas, strategies such as *count-on-from-larger, doubles and near-doubles,* and *make-to-ten* for addition, *think-of-addition, make-back-to-ten* for subtraction, and *doubles-and-one more-group* for multiplication), games and practice

INITIAL RECORDING (basic facts) to support place-value and avoid = sign

eg,	3	14	6	6
	+ 4	- 8	x7	4) 24

FORMAL RECORDING (beyond basic facts) using bundling/Base 10 materials, number expanders and mental strategies, written recording to support place-value

MENTAL COMPUTATION (beyond basic facts) using place-value ideas and extended mental strategies, thinking strings and open number lines.

EXTEND RECORDING to decimals, common fractions and larger numbers in ways which build on to prior knowledge and renaming strategies, eg,

4.26	5⅛
+ 7.38	- 3⅔

Think: eighths by thirds…twenty-fourths, rename in terms of 24 parts…think factors…8 parts is increased to 24 parts by a factor of 3, so 1/8 = 3/24…3 parts increased by a factor of 8, so 2/3 = 16/24 … decompose and rename a 1 as 24/24 …

DEVELOP MENTAL STRATEGIES FOR ADDITION & SUBTRACTION FACTS TO 20 (oral not written)

1. COUNT ON FROM LARGER (for combinations involving 1, 2 or 3)

eg,	seven and two (7 and 2)	"seven ... eight, nine"
	three and six (3 and 6)	"six ... seven, eight, nine"
	one and eight (1 and 8)	"eight ... nine"
	two and nine (2 and 9)	"nine ... ten, eleven"

2. DOUBLES & NEAR DOUBLES

eg,	4 and 5	"four and four is eight, one more, nine"
	8 and 7	"eight and eight, sixteen, one less, fifteen"
	6 and 7	"six and six is twelve, one more thirteen"
	8 and 9	"double eight, sixteen, one more, seventeen"

3. MAKE TO TEN & COUNT ON

eg,	4 and 7	"seven, ten, eleven"
	8 and 6	8 ... 10, and four more, 12
	7 and 5	7 ... 10, and two more, 12
	4 and 9	9 ... 10, and three more, 13

4. EXTEND STRATEGIES TO SUPPORT SUBTRACTION

eg,	9 take-away 3	"nine ..., eight, seven, six" (count-back, 1, 2 or 3 only)
	14 take-away 8	14 ... 7 (halving), 6 (halving)
	16 take-away 9	16 ... 10, (take three more), 7 (make back to ten)
	12 take-away 5	"think, 5 and what is 12?, 5 and 7" (think of addition)

5. DEVELOP STRATEGIES FOR RELATED FACTS

eg,	3 tens and 5 tens	"five..., six, seven, eight tens"
	14 tens take 8 tens	"8 and what is 14? … 6 tens"
	36 and 7	"43 because 6 and 7 is 13"

INTRODUCE INITIAL RECORDING AS STRATEGIES ARE ESTABLISHED

eg, $\begin{array}{r} 2 \\ +\ 7 \\ \hline \end{array}$ **Think:** 7… 8, 9 $\begin{array}{r} 6 \\ +\ 7 \\ \hline \end{array}$ **Think:** double 6, 12, and 1 more, 13

$\begin{array}{r} 6 \\ +\ 8 \\ \hline \end{array}$ **Think:** 8 … 10, and 4 more, 14

INTRODUCE FORMAL RECORDING TO SUPPORT PLACE-VALUE IDEAS
That is, by using appropriate materials such as Base 10 blocks and number expanders

DEVELOP RECORDING STRATEGIES TO SUPPORT MENTAL COMPUTATION

INTRODUCE AND EXPLORE THE USE OF 'THINKING STRINGS':

eg, 35 and 76? Record and analyse:

76, 106, 111	"you started with the larger, added 3 tens then 5 more"
35, 40, 41, 111	"you made to nearest ten, then added 1 more, then 7 tens"

11, 81, 111 "you added ones, then 7 tens and 3 tens more"

EXTEND RENAMING TO DEVELOP 'SAME AS' STRATEGY:

eg, 57 take-away 29?

SAME AS: 58 take-away 30.... 58 take 3 tens, 28

Advantage of single step, avoids multiple steps and tendency to add or subtract incorrectly when rounding numbers and adjusting at a later stage.

EXTEND VISUAL COUNTING STRATEGIES:

eg, three addends, no recording (inspection only)

24	56
72	sum?

6 ones and 4 ones is 1 ten
5 tens and 2 tens is 7 tens
so 8 tens altogether
7 more tens, 15 tens, 150
and 2 more ones is 152

Use the same idea for subtraction by including the total in the shaded area and omitting one of the addends.

eg, missing addend, no recording (inspection only)

43	17
?	132

132 take 1 ten is 122, take 7 is 120, 115
115 take 4 tens (THINK: 11 tens, take 4 tens is 7 tens), 75

75 take away 3 ones, 72

Can you think of another way?

INTRODUCE MULTIPLICATION & DIVISION IDEAS (through action stories and modelling)

- Use action stories involving *groups of* and *arrays* to explore concepts
- Model more efficient counting strategies, such as skip counting (small numbers only)
- Encourage children to invent and represent their own action stories.

BUILD MENTAL STRATEGIES FOR MULTIPLICATION AND DIVISION FACTS (based on the known strategies, arrays and regions)

1. The 2s facts

eg, 2 ones, 2 twos, 2 threes, 2 fours, ... 2 eights ....

DOUBLES, eg, "2 sevens ... double 7, 14"

Establish RELATED facts, eg, 7 twos, think 2 sevens

2. The 3s facts

eg, 3 ones, 3 twos, 3 threes, 3 fours, ... 3 eights, 3 nines ....

DOUBLES AND 1 MORE GROUP

eg, "3 eights ... double 8, 16 and 8 more, 20 ... 24"

Establish RELATED facts, eg, 8 threes, think 3 eights

3. The 4s facts

eg, 4 ones, 4 twos, 4 threes, 4 fours, ... 4 eights, 4 nines ....

DOUBLE DOUBLES

eg, "4 sixes ... double 6, 12, double 12, 24"

Establish RELATED facts, eg, 6 fours, think 4 sixes

4. The 5s facts

eg, ... 5 threes, 5 fours, 5 fives, 5 sixes .... 5 eights, 5 nines ....

RELATE TO TENS

eg, "5 eights is half of 10 eights, 40

Establish RELATED facts, eg, 8 fives, think 5 eights or 4 tens

OR RELATE TO READING TIME ON A CLOCKFACE

see 1, 2, 3, 4, ... read as 5, 10,15, 20 minutes past the hour

eg, "minute hand on 4 means 20 minutes past the hour

5. The 9s facts

eg, ... 9 sixes, 9 sevens, 9 eights, 9 nines

TEN GROUPS LESS 1 GROUP

eg, 9 eights is less than 10 eights, it is 8 less, 72"

Establish RELATED facts using the same strategy,

eg, 8 nines is less than 8 tens it is 8 less, 72

6. The 1s and 0s facts

eg, 1 one, 1 two, 1 three, 1 four ... 1 of anything is anything

Establish RELATED facts, eg, 8 ones, think 1 eight

eg, 0 ones, 0 twos, 0 threes, 0 fours,... 0 'anythings' is nothing

Establish RELATED facts, eg, 9 zeros, think 0 nines

7. Deal with remaining facts

x	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

6 sevens, 3 sevens and 3 sevens, 42

7 sevens, 49 SQUARE NUMBER

8 sixes, double 6, 12 double, double, 48

8. Consolidate and build to speed and accuracy via games such as 'Beat the Teacher' and 'Multiplication Toss'.

eg, Multiplication Toss

Equipment: 2 ten-sided dice and an A4 sheet of cm grid paper for each player.

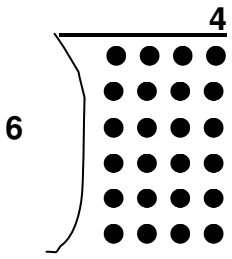
Rules: Two or more players take turns to toss 2 ten-sided dice (2 six-sided dice could be used initially). The result of the toss determines region possibilities, eg, a 6 and 4 could be recorded as a 6 by 4 rectangle on the player's A4 sheet (6 fours) or a 4 by 6 rectangle (4 sixes). The relevant fact is recorded in the region chosen. The Game proceeds with no regions overlapping until a player cannot take their turn in which case they either miss a turn or partition their region (see Extension below).

The winner is the player with the least number of uncovered cm squares.

Extension: If 8 and 6 are thrown but there is no room left on the grid to record 8 sixes or 6 eights, the turn can still be taken by partitioning once only, eg, 4 sixes and 4 sixes, or 5 sixes and 3 sixes, which could be played as 5 sixes and 6 threes for example.

EXTEND TO DIVISION FACTS USING ARRAY IDEA AND 'THINK OF MULTIPLICATION' STRATEGY

eg, 24 divided by 6



THINK: 6 what's are 24?

Relates to ARRAY idea and supports more efficient estimation or known-fact strategies rather than count-all groups or experimental skip counting.

REPRESENTING THE MULTIPLICATION AND DIVISION FACTS (read from left to right)

X	1	2	3	4	5	6	7	8	9
1	1 one 1	1 two 2	1 three 3	1 four 4	1 five 5	1 six 6	1 seven 7	1 eight 8	1 nine 9
2	2 ones 2	2 twos 4	2 threes 6	2 fours 8	2 fives 10	2 sixes 12	2 sevens 14	2 eights 16	2 nines 18
3	3 ones 3	3 twos 6	3 threes 9	3 fours 12	3 fives 15	3 sixes 18	3 sevens 21	3 eights 24	3 nines 27
4	4 ones 4	4 twos 8	4 threes 12	4 fours 16	4 fives 20	4 sixes 24	4 sevens 28	4 eights 32	4 nines 36
5	5 ones 5	5 twos 10	5 threes 15	5 fours 20	5 fives 25	5 sixes 30	5 sevens 35	5 eights 40	5 nines 45
6	6 ones 6	6 twos 12	6 threes 18	6 fours 24	6 fives 30	6 sixes 36	6 sevens 42	6 eights 48	6 nines 54
7	7 ones 7	7 twos 14	7 threes 21	7 fours 28	7 fives 35	7 sixes 42	7 sevens 49	7 eights 56	7 nines 63
8	8 ones 8	8 twos 16	8 threes 24	8 fours 32	8 fives 40	8 sixes 48	8 sevens 56	8 eights 64	8 nines 72
9	9 ones 9	9 twos 18	9 threes 27	9 fours 36	9 fives 45	9 sixes 54	9 sevens 63	9 eights 72	9 nines 81

RECORD USING CONSISTENT, MEANINGFUL LANGUAGE WHICH SUPPORTS PLACE-VALUE:

eg. For addition with regrouping

Hundreds	Tens	Ones
1	1	
3	**6**	**7**
+ 2	**4**	**9**
6	**1**	**6**

Add the ones: 7 ones and 9 ones
16 ones, rename as1 ten and 6 ones
Record the ten with the tens and the ones with the ones
Add the tens: 1 ten and 6 tens and 4 tens
11 tens, rename: 1 hundred and 1 ten
Record the hundred with the hundreds and the tens with the tens
Add the hundreds: 1 hundred and 3 hundreds and 2 hundreds, 6 hundreds

eg. For subtraction with trading

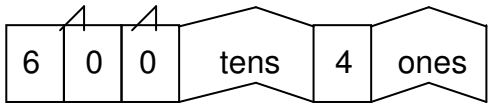
Hundreds	Tens	Ones
	12	
7	~~2~~	12
8	**~~3~~**	**~~2~~**
- 4	**5**	**7**
3	**7**	**5**

Can I take 7 ones from 2 ones? NO
Trade 1 ten for 10 ones, 12 ones
12 ones take 7 ones is 5 ones
Record with the ones
Can I take 5 tens from 2 tens? NO
Trade 1 hundred for 10 tens, 12 tens
12 tens take 5 tens is 7 tens
Record with the tens
Can I take 4 hundreds from 7 hundreds?
YES, 7 hundreds take 4 hundreds is 3 hundreds.
Record with the hundreds

Try this:

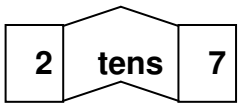
Th	H	Tens	Ones
5	9	9	14
~~6~~	**0**	**0**	**~~4~~**
- 2	**5**	**4**	**7**

Use a number expander:



Rename as 599 tens and 14 ones

eg. For multiplication with regrouping



ones

2 **X** 4

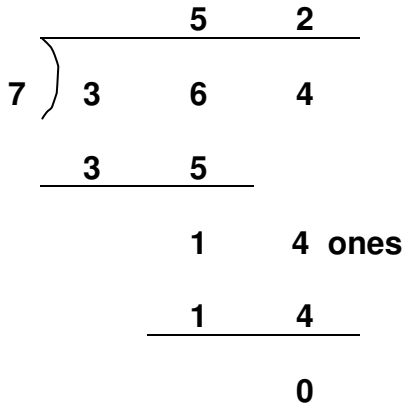
4 sevens are 28, record 8 ones and 2 tens to regroup
4 by 2 tens is 8 tens and 2 more tens is 10 tens, that is, 108

eg.

Th	H	Tens	Ones
	1	3	
	3	**2**	**4**
	x		**8**
2	**5**	**9**	**2**

8 by 4 ones is 32, 3 tens and 2 ones
Record the ones with the ones and the ten with the tens
8 ones by 2 tens is 16 tens and 3 more tens is 19 tens, 1 hundred and 9 tens
Record the tens with the tens and the hundreds with the hundreds
8 ones by 3 hundreds is 24 hundreds and 1 more hundred is 25 hundreds, 2 thousand and 5 hundreds
Record the hundreds with the hundreds and the thousands with the thousands

eg. For division with trading and no remainder



364 shared among 7
Can I share 3 hundreds so that everyone gets some hundreds? NO
Trade 3 hundreds for 30 tens, 36 tens
Can I share 36 tens so that everyone gets some tens? YES, 5 tens each
Record with the tens
How many tens have I shared? 35 tens
How many tens have I left to share? 1 ten
Can I share this? NO
Trade 1 ten for 10 ones, 14 ones
Can I share 14 ones so that everyone gets some ones? YES, 2 ones each
Record with the ones
How many ones have I shared? 14
Have I any left to share? NO

REMEMBER - IF YOU NEED OR WANT AN ANSWER - USE A CALCULATOR, MENTAL COMPUTATION OR ESTIMATION

WRITTEN COMPUTATION WILL DEVELOP WITH UNDERSTANDING AND PRACTICE - IT IS AN ESSENTIAL FOUNDATION OF MATHEMATICAL LITERACY

SOME USEFUL REFERENCES:

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